Solvability of dissipative second order left-invariant differential operators on the Heisenberg group

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Abstract

We prove local solvability for large classes of operators of the form

$$L = \sum_{j,k=1}^{2n} a_{jk} V_j V_k + i\alpha U,$$

where the V_j are left-invariant vector fields on the Heisenberg group satisfying the commutation relations $[V_j, V_{j+n}] = U$ for $1 \le j \le n$, and where $A = (a_{jk})$ is a complex symmetric matrix with semi-definite real part. Our results widely extend all of the results for the case of non-real, semi-definite matrices A known to date, in particular those obtained recently jointly with F. Ricci under Sjöstrand's cone condition. They are achieved by showing that an integration by parts argument, which had been applied in different forms in previous articles, ultimately allows for a reduction to the case of operators L whose associated Hamiltonian has a purely real spectrum. Various examples are given in order to indicate the potential scope of this approach and to illuminate some further conditions that will be introduced in the article. [1]

1 Introduction

Consider the standard basis of left-invariant vector fields on the Heisenberg group $\mathbb{H}_n$, with coordinates $(x, y, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$:

$$\begin{aligned} X_j &= \partial_{x_j} - \frac{1}{2} y_j \partial_u \ , \qquad j = 1, \cdots, n \ , \\ Y_j &= \partial_{y_j} + \frac{1}{2} x_j \partial_u \ , \qquad j = 1, \cdots, n \ , \\ U &= \partial_u. \end{aligned} \tag{1.1}$$

We also write $V_1, \dots, V_{2n}$ for $X_1, \dots, X_n, Y_1, \dots, Y_n$ (in this order), and, consistently, $v = (x, y) \in \mathbb{R}^{2n}$.

Given a $2n \times 2n$ complex symmetric matrix $A = (a_{jk})$, set

$$\mathcal{L}_A = \sum_{j,k=1}^{2n} a_{jk} V_j V_k \ , \tag{1.2}$$

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and, for $\alpha \in \mathbb{C}$,

$$\mathcal{L}_{A,\alpha} = \mathcal{L}_A + i\alpha U \ . \tag{1.3}$$

These operators can be characterized as the second order left-invariant differential operators on $\mathbb{H}_n$ which are homogeneous of degree 2 under the automorphic dilations $(v, u) \mapsto (\delta v, \delta^2 u)$.

It is the goal of this article to devise large classes of operators $\mathcal{L}_{A,\alpha}$ with non-real coefficient matrices A that are locally solvable, extending and unifying in this way results from previous articles, including [3], [9], [10] and [11]. The case of real coefficients had been treated in a complete way in [13], see also [15]. For background information on the problem of local solvability for such operators and further references, see [14], [11].

In order to motivate the conditions on the coefficient matrix A that we shall impose, let me point out that there are various indications that, at least in "generic" situations of sufficiently high dimension, local solvability of $\mathcal{L}_{A,\alpha}$ will only occur if the principal symbol p_A satisfies a sign condition, i.e if there exists some $\theta \in \mathbb{R}$ such that $\mathrm{Re}\,(e^{i\theta} p_A) \geq 0$ (see e.g. [10], [8] and the examples to follow). We know of locally solvable operators $\mathcal{L}_{A,\alpha}$ which do not satisfy a sign condition (see [7], [8]), but these examples effectively only occur in low dimensions. For $\theta = 0$ (which we may then assume without loss of generality), the sign condition means that $\mathcal{L}_{A,\alpha}$ is a dissipative differential operator, or, equivalently, that $\mathrm{Re}\,A \geq 0$. The latter condition is what we shall assume throughout the paper. This condition is considerably weaker than Sjöstrand's cone condition (see [17]), which was imposed in [11] and which for the operators (1.3) just means that there is a constant $C > 0$ such that $|\mathrm{Im}\,A| \leq C\,\mathrm{Re}\,A$.

The operators $\mathcal{L}_{A,\alpha}$ have double characteristics, and for such operators it is known that it is not only the principal symbol that governs local solvability, but that also the subprincipal symbol in combination with the Hamiltonian mappings associated with doubly characteristic points plays an important role. Due to the translation invariance of our operators and the symplectic structure that is inherent in the Heisenberg group law, these Hamiltonians are essentially encoded in the Hamiltonian $S \in \mathfrak{sp}(n, \mathbb{C})$, associated to the coefficient matrix A by the relation $S := -AJ$ (see e.g. [14]). Here, J denotes the matrix $J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. In order to emphasize the central role played by S, we shall therefore also denote $\mathcal{L}_{A,\alpha}$ by $L_{S,\alpha}$.

One of our main results, Theorem 2.1, states that, under a further, natural condition, the question of local solvability of the operators $L_{S,\alpha}$ can essentially be reduced to the case where the Hamiltonian S has only real eigenvalues.

This is achieved by showing that an integration by parts technic, which had been introduced by R. Beals and P.C. Greiner in [1] and since then been applied in modified ways in various subsequent articles, e.g. in [11], when viewed in the right way, ultimately allows to show that $L_{S,\alpha}$ is locally solvable, provided that $L_{S_r,\beta}$ is locally solvable for particular values of β. Here, S_r is the "part" of S comprising all Jordan blocks associated with real eigenvalues.

In combination with some partial results for the case of real eigenvalues, this theorem allows to widely extend all the results known to date for operators $\mathcal{L}_{A,\alpha}$ with non-real coefficient matrices A (see Theorems 2.5, 2.7). Moreover, we believe that our proofs have become simplier compared e.g. to [11], because of the new structural insights given by Theorem 2.1.

The article is organized as follows. Section 2 contains the basic notation, which is mostly taken from [11], and the statements of the main results. Moreover, we present various examples which help to illustrate some additional conditions imposed in our theorems.

The preparatory material on the algebraic properties of symplectic transformations needed in the proofs of our main results is collected in Section 3.

Section 4 is devoted to the discussion of examples, including those mentioned in Section 2.

In Section 5, following some line of thoughts in [1], we derive explicit formulas for the one-parameter semigroups $\{e^{t\mathcal{L}_{A,\alpha}}\}_{t>0}$. Since $\mathcal{L}_{A,\alpha}$ is assumed to be dissipative, these semigroups do exist, at least as linear contractions on $L^2(\mathbb{H}_n)$. For the case where $\operatorname{Re} A$ is strictly positive definite, such formulas had been established in [11] by means of the oscillator semigroup, introduced by R. Howe in [6]. Starting from formula (5.10) in Theorem 5.2, [11], we extend its range of validity to arbitrary matrices A with $\operatorname{Re} A \geq 0$, by adapting some limit arguments from [5] to the setting of twisted convolution operators. We should like to mention that the main result in this section, Theorem 5.5, could also be derived directly from Theorem 4.3 in [5] by means of the Weyl transform, which relates twisted convolution operators to pseudo-differential operators in the Weyl calculus. We prefer to present our approach, nevertheless, since we believe that the approach through the oscillator semigroup is somewhat simpler than in [5] and [1].

We also study the analytic extension of our formulas to the case of arbitrary complex matrices A and complex time parameter t. This will be useful in the proof of Theorem 2.1, which will be given Section 6, in that it allows to use complex symplectic changes of coordinates in some situations.

Finally, in Section 7, we prove some partial results on the case where S has purely real spectrum, which in combination with results from [13] lead to Theorem 2.5. Moreover, in Proposition 7.3, we shall prove by means of a representation theoretic criterion from [8] that the operators from Example 2.6 are always locally solvable. This result shows that the simple minded approach used in Proposition 7.1 is rather limited, and that new ideas will be need in order to obtain a better understanding of the case where S has purely real spectrum.

2 Statement of the main results

In order to emphasize the symplectic structure on $\mathbb{R}^{2n}$ which is implicit in (1.1), and at the same time to provide a coordinate-free approach, we shall adopt the notation from [11] and work within the setting of an arbitrary $2n$-dimensional real vector space V, endowed with a symplectic form σ. The extension of σ to a complex symplectic form on $V^{\mathbb{C}}$, the complexification of V, will also be denoted by σ.

If Q is a complex-valued symmetric form on V, we shall often view it as a symmetric bilinear form on $V^{\mathbb{C}}$, and shall denote by $Q(v)$ the quadratic form $Q(v,v)$. Q and σ determine a linear endomorphism S of $V^{\mathbb{C}}$ by imposing that

$$\sigma(v, Sw) = Q(v, w).$$

Then, $S \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma)$, i.e.

$$\sigma(Sv, w) + \sigma(v, Sw) = 0. \tag{2.4}$$

S is called the *Hamilton map* of V. Clearly, S is real, i.e. $S \in \mathfrak{sp}(V, \sigma)$, if Q is real.

If $T : U \to W$ is a linear homomorphism of real or complex vector spaces, we shall denote by ${}^tT : W^* \to U^*$ the transposed homomorphisms between the dual spaces W^* and U^* of W and U, respectively, i.e.

$$({}^tTw^*)(u) = w^*(Tu), \; u \in U, w^* \in W^*.$$

As usually, we shall identify the bi-dual W^{**} with W.

If Q is any bilinear form on V (respectively $V^{\mathbb{C}}$), there is a unique linear map $\mathcal{Q} : V \to V^*$ (respectively $\mathcal{Q} : V^{\mathbb{C}} \to (V^{\mathbb{C}})^*$) such that

$$(\mathcal{Q}v)(w) = Q(v, w), \tag{2.5}$$

and $\mathcal{Q}$ is a linear isomorphism if and only if Q is non-degenerate. In particular, the map $\mathcal{J} : V^{\mathbb{C}} \to (V^{\mathbb{C}})^*$, given by

$$(\mathcal{J}v)(w) = \sigma(w, v), \tag{2.6}$$

is a linear isomorphism, which restricts to a linear isomorphisms from V to V^*, also denoted by $\mathcal{J}$. We have ${}^t\mathcal{J} = -\mathcal{J}$, so that (2.4) can be read as

$$\mathcal{J}S + {}^tS\mathcal{J} = 0. \tag{2.7}$$

Moreover, if the form Q in (2.5) is symmetric, then ${}^t\mathcal{Q} = \mathcal{Q}$ and $\mathcal{Q} = \mathcal{J}S$.

By composition with $\mathcal{J}$, bilinear forms on V can be transported to V^*, e.g. we put

$$\sigma^*(\mathcal{J}v, \mathcal{J}w) := \sigma(v, w), \quad Q^*(\mathcal{J}v, \mathcal{J}w) := Q(v, w).$$

In analogy with (2.5) and (2.6), we obtain maps from V^* to V (respectively from $(V^{\mathbb{C}})^*$ to $V^{\mathbb{C}}$) which satisfy the following identities:

$$\mathcal{J}^* = -\mathcal{J}^{-1}, \; \mathcal{Q} = {}^t\mathcal{J}\mathcal{Q}^*\mathcal{J} = -\mathcal{J}\mathcal{Q}^*\mathcal{J}, \; S^* = \mathcal{J}S\mathcal{J}^{-1} = -{}^tS. \tag{2.8}$$

The canonical model of a symplectic vector space is $\mathbb{R}^{2n}$, with symplectic form $\sigma(v, w) = {}^tvJw =: \langle v, w \rangle$, where

$$J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Identifying also the dual space with $\mathbb{R}^{2n}$ (via the canonical inner product on $\mathbb{R}^{2n}$), we have $\mathcal{J}v = Jv$. Moreover, of course $(\mathbb{R}^{2n})^{\mathbb{C}} = \mathbb{C}^{2n}$.

If a general symmetric form Q is given by $Q(v, w) = {}^tvAw$, where A is a symmetric matrix, we have the following formulas:

$$\mathcal{Q}v = Av, \; Sv = -JAv, \; S^*v = -AJv. \tag{2.9}$$

These formulas apply whenever we introduce coordinates on V adapted to a symplectic basis of V, i.e. to a basis $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ such that

$$\sigma(X_j, X_k) = \sigma(Y_j, Y_k) = 0, \; \sigma(X_j, Y_k) = \delta_{jk}$$

for every j, k. Observe that in V^*, the dual of a symplectic basis is symplectic with respect to σ^*. The Heisenberg group $\mathbb{H}_V$ built on V is $V \times \mathbb{R}$, endowed with the product

$$(v, u)(v', u') := (v + v', u + u' + \frac{1}{2}\sigma(v, v')).$$

Its Lie algebra $\mathfrak{h}_V$ is generated by the left-invariant vector fields

$$X_v = \partial_v + \frac{1}{2}\sigma(\cdot, v)\partial_u, \quad v \in V.$$

The Lie brackets are given by $[X_v, X_w] = \sigma(v, w)U$, with $U := \partial_u$.

We regard the formal expression (1.2) defining the operator $\mathcal{L}_A$ as an element of the symmetric tensor product $\mathfrak{S}^2(V^{\mathbb{C}})$ (with $V^{\mathbb{C}} = \mathbb{C}^{2n}$), hence as a complex symmetric bilinear form Q^* on $(V^{\mathbb{C}})^*$. With this notation, the Hamilton map S^* of Q^* is

$$S^*v = -JAv, \tag{2.10}$$

and the Hamilton map of the corresponding form Q on $V^{\mathbb{C}}$ is

$$Sv = -AJv. \tag{2.11}$$

Since the solvability of $\mathcal{L}_{A,\alpha}$ is closely connected to the spectral properties of the associated Hamilton map, we shall also write

$$\mathcal{L}_A =: L_S, \quad \mathcal{L}_{A,\alpha} =: L_{S,\alpha}.$$

We remark that $[L_{S_1}, L_{S_2}] = -2\ L_{[S_1,S_2]}U$. The following structure theory for elements $S \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma)$ will be important (see e.g. [5], [11]).

If $\operatorname{spec} S \subset \mathbb{C}$ denotes the spectrum of S, then $-\lambda \in \operatorname{spec} S$ whenever $\lambda \in \operatorname{spec} S$. Moreover, if V_λ denotes the generalized eigenspace of S belonging to the eigenvalue λ, then

$$\sigma(V_\lambda, V_\mu) = 0, \quad \text{if } \lambda + \mu \neq 0. \tag{2.12}$$

In particular, V_λ and $V_{-\lambda}$ are isotropic subspaces with respect to the symplectic form σ, and $V_\lambda \oplus V_{-\lambda}$ is a symplectic subspace of $V^{\mathbb{C}}$, if $\lambda \neq 0$, while V_0 is symplectic too. We thus obtain a decomposition of $V^{\mathbb{C}}$ as a direct sum of symplectic subspaces which are σ-orthogonal:

$$V^{\mathbb{C}} = V_0 \oplus \sum_{\lambda \neq 0}^{\oplus} V_\lambda \oplus V_{-\lambda}. \tag{2.13}$$

Here, the summation takes place over a suitable subset of $\operatorname{spec} S$. Notice that the decomposition above is also orthogonal with respect to the symmetric form $Q(v, w) = \sigma(v, Sw)$, since the spaces V_λ are S-invariant. (2.13) induces an orthogonal decomposition

$$V^{\mathbb{C}} = V_r \oplus V_i, \tag{2.14}$$

where $V_r := \sum^{\oplus}_{\lambda \in \mathbb{R} \cap \operatorname{spec} S} V_\lambda$ and $V_i := \sum^{\oplus}_{\mu \in (\mathbb{C} \setminus \mathbb{R}) \cap \operatorname{spec} S} V_\mu$. Correspondingly, S decomposes as

$$S = S_r + S_i, \tag{2.15}$$

where we have put $S_r(u+w) := S(u)$, $S_i(u+w) := S(w)$, if $u \in V_r$ and $w \in V_i$. Then also S_r and S_i are in $\mathfrak{sp}(V^{\mathbb{C}}, \sigma)$, and S_r respectively S_i corresponds to the Jordan blocks of S associated with real eigenvalues respectively non-real eigenvalues.

Next, S can be uniquely decomposed as $S = D + N$ such that D is semisimple, N is nilpotent and $DN = ND$. The endomorphisms D and N in this Jordan decomposition of S are polynomials in S, and it is known from general Lie theory that $D, N \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma)$ (see e.g. [2]).

This can also be seen directly. For this purpose, we may assume without loss of generality that $V^{\mathbb{C}} = V_\lambda \oplus V_\lambda$, for some $\lambda \neq 0$ (the case $V^{\mathbb{C}} = V_0$ is obvious). Then, if v_λ, w_λ are in V_λ and $v_{-\lambda}, w_{-\lambda}$ in $V_{-\lambda}$, we have

$$\begin{aligned}\sigma(D(v_\lambda + v_{-\lambda}), w_\lambda + w_{-\lambda}) &= \sigma(\lambda v_\lambda - \lambda v_{-\lambda}, w_\lambda + w_{-\lambda}) \\ &= \sigma(v_\lambda, \lambda w_{-\lambda}) + \sigma(v_{-\lambda}, -\lambda w_\lambda) \\ &= -\sigma(v_\lambda + v_{-\lambda}, D(w_\lambda + w_{-\lambda})).\end{aligned}$$

Applying the Jordan decomposition to S_r, we can uniquely write

$$S_r = D_r + N_r, \tag{2.16}$$

with D_r semisimple, N_r nilpotent and $D_r N_r = N_r D_r$.

Our first main result is a reduction theorem, which allows in many cases to reduce the question of local solvability of $L_{S,\alpha}$ to essentially the same question for the operator $L_{S_r,\beta}$, for particular values of $\beta \in \mathbb{C}$. Its proof is based on an integration by parts argument, variants of which had already been used in [1] as well as in several subsequent articles, e.g. in [9], [11]. We believe that our approach reveals more clearly and conceptually the potential range of validity of such technics of integration by parts, by showing that they allow a reduction to the study of the operators $L_{S_r,\beta}$.

A main obstruction to applying this technic is the fact that the spaces $V_\lambda \oplus V_{-\lambda}$, for $\lambda \in \mathbb{R} \setminus \{0\}$, are in general not invariant under complex conjugation. This had already been observed by L. Hörmander [5], who gave an example for the related case $\lambda = 0$, and we shall give further examples in Section 4.

We shall therefore mostly work under the following hypothesis:

(R) $\quad V_\lambda \oplus V_{-\lambda}$ is invariant under complex conjugation, for every $\lambda \in \mathbb{R} \setminus \{0\}$.

We write

$$\operatorname{spec} S_i = \{\pm\omega_1, \ldots, \pm\omega_{n_1}\} \subset \mathbb{C} \setminus \mathbb{R},$$

where the eigenvalues $\pm\omega_j$ are listed with their multiplicities, and where

$$\nu_j := \operatorname{Im} \omega_j > 0, \ j = 1, \ldots, n_1. \tag{2.17}$$

We also put

$$\nu := \sum_{j=1}^{n_1} \nu_j, \quad \nu_{\min} := \min_{j=1,\ldots,n_1} \nu_j \ .$$

Theorem 2.1 *Assume that* $\operatorname{Re} Q_S \geq 0$, *that* $S_i \neq 0$ *(i.e.* $\nu > 0$*), and that condition (R) is satisfied. Then, the following holds:*

(i) *$L_S + i\alpha U$ is locally solvable (and even admits a tempered fundamental solution), if $|\mathrm{Re}\,\alpha| < \nu$.*

(ii) *If $M > 0$, and if $|\mathrm{Re}\,\alpha| < M$, then $L_S + i\alpha U$ is locally solvable, provided that $L_{S_r} + i(\alpha \pm \sum_{j=1}^{n_1}(2k_j+1)i\omega_j)U$ is locally solvable, for every n_1-tupel $(k_1, \ldots, k_{n_1}) \in \mathbb{N}^{n_1}$ such that $\sum_{j=1}^{n_1}(2k_j+1)\nu_j < M$.*

In view of Theorem 2.1, it thus becomes a major task to understand local solvability when S has purely real spectrum.

There are various indications that, at least in sufficiently high dimensions, $L_{S,\alpha}$ may not be locally solvable, unless S satisfies the following sign condition:

$$\mathrm{Re}\,(e^{i\theta}Q_S) \geq 0 \quad \text{for some } \theta \in \mathbb{R} \tag{2.18}$$

(see e.g. [8], and also the examples to follow). We shall therefore assume that (2.18) holds for S_r, and even that, without loss of generality, $\mathrm{Re}\,Q_{S_r} \geq 0$. The following proposition gives a sufficient condition for this to hold.

Proposition 2.2 *Assume that property (R) is satisfied and that $N_r^2 = 0$, where N_r denotes the nilpotent part in the Jordan decomposition of S_r. Then $\mathrm{Re}\,Q_S \geq 0$ implies $\mathrm{Re}\,Q_{S_r} \geq 0$.*

Consider the following examples, which shed some more light on the conditions in Proposition 2.5.

Example 2.3 On $\mathbb{H}_3$, consider

$$L_S := Y_1^2 + X_3^2 + 2X_3Y_1 + Y_3^2 + 2i(X_1Y_2 - X_2Y_1 - Y_2Y_3).$$

It is clear that $\mathrm{Re}\,Q_S \geq 0$. However, we will show in Section 4 that neither $\mathrm{Re}\,Q_{S_r} \geq 0$, nor $\mathrm{Re}\,Q_{S_i} \geq 0$, even though N_r is 2-step nilpotent. Even worse, by Hörmander's criterion (H), one checks that $L_{S_r} + F$ und also $L_{S_i} + F$ are not locally solvable, for every first order differential operator F with smooth coefficients. This shows that Proposition 2.2 fails to be true without property (R), and that the entire approach in Theorem 2.1 will in general break down, if (R) is not satisfied. Theorem 2.1 (i) remains nevertheless valid in this example, i.e. $L_S + i\alpha U$ is locally solvable for $|\mathrm{Re}\,\alpha| < 1$ (see Remark 6.3 following Proposition 6.1).We do not know what happens if $|\mathrm{Re}\,\alpha| \geq 1$.

Example 2.4 On $\mathbb{H}_3$, consider

$$L_S := X_2^2 + X_3^2 + Y_3^2 + 2i(X_1Y_2 + bX_2Y_3), \quad b \in \mathbb{R} \setminus \{0\}.$$

Again, we have $\mathrm{Re}\,Q_S \geq 0$. We shall see that $S_r = N_r$ is nilpotent of step 4 in this example, and that $\mathrm{Re}\,Q_{S_r} \geq 0$ is not satisfied (not even (2.18)). Again, Hörmander's condition (H) is satisfied by L_{S_r}, so that $L_{S_r} + F$ is not locally solvable, for every first order term F. Notice that in this example property (R) does hold, since $S_r = N_r$, which shows that the conclusion of Proposition 2.5 will in general not hold, if N_r is nilpotent of step higher than 2, even under property (R).

In our study of L_{S_r}, we shall therefore restrict ourselves to the case where $N_r^2 = 0$.

Let us thus assume, for a moment, that $\operatorname{spec} S \subset \mathbb{R}$, i.e. $S = S_r$, and that $N^2 = 0$, where $S = D + N$ is the Jordan decomposition of S. We write $D = D_1 + iD_2$, where $D_1 = \operatorname{Re} D$, $D_2 = \operatorname{Im} D$. If we assume that D satisfies the following hypothesis

$$\text{(C)} \qquad [D_1, D_2] = 0,$$

then we can discuss local solvability of $L_S + i\alpha U$ in a complete way, even without assuming (R), by means of certain a priori estimates and the results in [13].

Theorem 2.5 *Assume that $S \neq 0$ has purely real spectrum, $\operatorname{Re} Q_S \geq 0$, $N^2 = 0$, and that property (C) is satisfied. Then the following holds true:*

(i) If $\operatorname{Re} S \neq 0$ or $\operatorname{Re} \alpha \neq 0$, then $L_S + i\alpha U$ is locally solvable.

(ii) If $\operatorname{Re} S = 0$, $\operatorname{Re} \alpha = 0$ and $N \neq 0$, then $L_S + i\alpha U$ is locally solvable.

(iii) If $\operatorname{Re} S = 0$, $\operatorname{Re} \alpha = 0$ and $N = 0$ then, after applying a suitable automorphism of $\mathbb{H}_n$ leaving the center fixed, L_S takes the form

$$L_S = \sum_{j=1}^{n} i\lambda_j (X_j^2 + Y_j^2), \tag{2.19}$$

with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. In particular, $\operatorname{spec} S = \{\pm\lambda_1, \ldots, \pm\lambda_n\}$. Then $L_S + i\alpha U$ is locally solvable if and only if there are constants $C > 0$ and $M \in \mathbb{N}$, such that

$$|\alpha \pm \sum_{j=1}^{n} (2k_j + 1) i\lambda_j| \geq C(1 + |k|)^{-M}, \tag{2.20}$$

for every $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$.

Consider the following example.

Example 2.6 On $\mathbb{H}_2$, let

$$L_S := (m + c_1)Y_1^2 + (m - c_1)Y_2^2 + 2c_2 Y_1 Y_2 + 2i(X_1 Y_2 - X_2 Y_1),$$

where we assume that $c_1^2 + c_2^2 \neq 0$. It is easy to see that for $m \geq \sqrt{c_1^2 + c_2^2}$ one has $\operatorname{Re} Q_S \geq 0$. Moreover, we shall show that D and N are given by the block matrices

$$D = \begin{pmatrix} iJ & 0 \\ C & iJ \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ mI & 0 \end{pmatrix},$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} c_1 & c_2 \\ c_2 & -c_1 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and that $D^2 = I$, so that $N^2 = 0$ and $\operatorname{spec} S = \{-1, 1\}$. One checks immediately that property (C) is not satisfied in this example. Nevertheless, we shall prove in Section 7, Proposition 7.3, by means of some explicit computations based on the group Fourier transform on $\mathbb{H}_2$, that $L_S + i\alpha U$ is locally solvable, for every $\alpha \in \mathbb{C}$.

This example indicates that condition (C) may not be necessary in Theorem 2.5. We shall further comment on Example 2.6 in Section 4.

The conditions in Theorem 2.5 are of course rather restrictive, but at present we do not know of any approach which would allow to discuss much wider classes of operators $L_S + i\alpha U$, with $\alpha \in \mathbb{C}$ and $\operatorname{spec} S \subset \mathbb{R}$, $\operatorname{Re} Q_S \geq 0$, even if S is nilpotent. A first, useful step towards a better understanding of such operators might be a classification of normal forms of matrices S satisfying $S^2 = 0$, along the lines of [16].

Nevertheless, Theorem 2.5 in combination with Theorem 2.1 immediately gives the subsequent theorem. It contains and widely extends, in combination with Theorem 2.5, all of the positive results on local solvability which have been obtained hitherto under the sign condition (2.18) in the "non-real" case (but for the discussion of the exceptional values arizing in (ii)).

Theorem 2.7 *Let $S \in \mathfrak{sp}(n, \mathbb{C})$ be such that $\operatorname{Re} Q_S \geq 0$. Assume further that S has at least one non-real eigenvalue and satisfies property (R), and that $N_r^2 = 0$ and $\operatorname{Re} D_r = 0$, where $S_r = D_r + N_r$ is the Jordan decomposition of S_r. Let $\omega_1, \ldots, \omega_{n_1}$ be as in Theorem 2.1. Then the following holds:*

(i) $L_S + i\alpha U$ is locally solvable for every $\alpha \in \mathbb{C}$, provided $S_r \neq 0$.

(ii) If $S_r = 0$, then $L_S + i\alpha U$ is locally solvable for all values of α in $\mathbb{C} \setminus \mathcal{E}_S$, where $\mathcal{E}_S$ is the following set of exceptional values:

$$\mathcal{E}_S := \{\pm \sum_{j=1}^{n_1} (2k_j + 1) i\omega_j : k_1, \ldots, k_{n_1} \in \mathbb{N}\}.$$

Remarks 2.8 (a) Assume that L_S satisfies the cone condition in the sense of Sjöstrand and Hörmander, i.e. there exists a constant $C > 0$, such that

$$|\operatorname{Im} Q_S(v)| \leq C |\operatorname{Re} Q_S(v)| \quad \forall v \in \mathbb{R}^{2n}. \tag{2.21}$$

Then it is known that S has at most one real eigenvalue, namely 0, that $S_r = N_r$ is 2-step nilpotent, and that $S_r = 0$ if and only if $\ker S$ is symplectic (see [11], Lemma 3.6).

Thus, Theorem 2.7 contains Theorem 2.2 in [11], except for the proof of non-solvability for the exceptional values $\alpha \in \mathcal{E}_S$, in case that $S_r = 0$.
(b) In the presence of property (R), the seemingly stronger condition $\operatorname{Re} D_r = 0$ is in fact equivalent to the condition (C) for S_r, i.e. $[\operatorname{Re} D_r, \operatorname{Im} D_r] = 0$ (see Corollary 3.7).

3 On the algebraic structure of S

Let $S \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma)$ be the Hamilton map associated to the quadratic form Q on V, which consequently we sometimes will also denote by Q_S. Our general assumption is that

$$\operatorname{Re} Q \geq 0 \text{ on } V. \tag{3.1}$$

This condition is equivalent to the following condition on the $\mathbb{C}$-linear extension of Q to $V^{\mathbb{C}} \times V^{\mathbb{C}}$:

$$\operatorname{Re} Q(z, \overline{z}) \geq 0 \quad \forall z \in V^{\mathbb{C}}. \tag{3.2}$$

Here, complex conjugation in $V^{\mathbb{C}}$ is meant with respect to the real form V of $V^{\mathbb{C}}$, i.e. for $z = v + iw \in V^{\mathbb{C}}$ we set $\overline{z} := v - iw$.

In the sequel, we shall often indicate the real part of a linear map or form by an index 1, the imaginary part by an index 2. For instance,

$$S = S_1 + iS_2, \quad Q = Q_1 + iQ_2.$$

The following proposition is due to Hörmander ([5], Proposition 4.4).

Proposition 3.1 *Assume that* $Q_1 = \operatorname{Re} Q \geq 0$. *Then the following hold true for every real eigenvalue* λ *of* S*:*

$$\overline{\operatorname{Ker}(S-\lambda)} = \operatorname{Ker}(S+\lambda); \tag{3.3}$$

$$S_1 \operatorname{Ker}(S \pm \lambda) = 0. \tag{3.4}$$

In particular, $\operatorname{Ker}(S-\lambda) \oplus \operatorname{Ker}(S+\lambda)$, $0 \neq \lambda \in \mathbb{R}$, *is the complexification of its intersection with* V, *and so is* $\operatorname{Ker} S$.

In [5] Hörmander gives an example which shows that the space $V_\lambda + V_{-\lambda}$, $\lambda \in \mathbb{R}$, is in general not self-conjugate, namely for the case $\lambda = 0$. We remark that our Example 2.3 presents a corresponding example for the case $\lambda = 1$. The following result is obvious.

Lemma 3.2 *Assume that* $\lambda \in \mathbb{R} \setminus 0$, *and that* $V_\lambda \oplus V_{-\lambda}$ *is self-conjugate. Then* $V_\lambda^{\mathbb{R}} := V \cap (V_\lambda \oplus V_{-\lambda})$ *is an* S_1 *and* S_2*-invariant real subspace of* V *such that* $V_\lambda \oplus V_{-\lambda} = V_\lambda^{\mathbb{R}} + iV_\lambda^{\mathbb{R}}$, *and* $\operatorname{Re} Q \geq 0$ *on* $V_\lambda^{\mathbb{R}}$, *if* (3.1) *holds. In particular,* $\operatorname{Re} Q \geq 0$ *on* $\sum\limits_{\lambda \in \mathbb{R}\setminus\{0\}} V_\lambda^{\mathbb{R}}$, *if* S *satisfies condition* (R).

Lemma 3.3 *Assume that condition* (3.1) *holds. Then* $\operatorname{Re} Q_{S_r} \geq 0$ *if and only if* $\overline{S(V_r)} \subset V_r$.

Proof. We have $\operatorname{Re} Q_{S_1} \geq 0$ if and only if

$$\operatorname{Re} \sigma(\overline{v}, S_r v) \geq 0 \quad \forall v \in V^{\mathbb{C}}.$$

Let $v = u + w, u \in V_r, w \in V_i$. Then

$$\operatorname{Re} \sigma(\overline{v}, S_r v) \;\; = \;\; \operatorname{Re} \sigma(\overline{u} + \overline{w}, Su) = \operatorname{Re} Q_S(u, \overline{u}) + \operatorname{Re} \sigma(\overline{w}, Su).$$

This is non-negative for every $v \in V^{\mathbb{C}}$ if and only if $\operatorname{Re} \sigma(\overline{w}, Su) = 0$ for every $u \in V_r, w \in V_i$. Since V_r and V_i are complex vector spaces, this means that $\sigma(\overline{w}, z) = 0$ or, equivalently, $\sigma(w, \overline{z}) = 0$ for every $w \in V_i$ and $z \in S(V_r)$. V_i being the orthogonal complement of V_r (w.r. to σ), the latter condition is equivalent to $\overline{S(V_r)} \subset V_r$.

Q.E.D.

We can now easily prove Proposition 2.2. Since $V_r = V_0 \oplus \sum\limits_{\lambda \in \mathbb{R}\setminus\{0\}} (V_\lambda \oplus V_{-\lambda})$, where each of the occuring subspaces is S-invariant, and where $\overline{V_\lambda \oplus V_{-\lambda}}$ is self-conjugate for $\lambda \neq 0$ (because of property (R)), we only have to show that $\overline{S(V_0)} \subset V_0$, in order to apply Lemma 3.3. But, $S(V_0) = N_r(V_0) \subset \operatorname{Ker} N_r \cap V_r = \operatorname{Ker} S$, and $\overline{\operatorname{Ker} S} = \operatorname{Ker} S$ by Proposition 3.1, so that $\overline{S(V_0)} \subset \operatorname{Ker} S \subset V_0$, which completes the proof.

Q.E.D.

The form $\operatorname{Re} Q_{N_r}$ is always semi-definite, if $N_r^2 = 0$, as the following result shows.

Lemma 3.4 *If $N_r^2 = 0$, then (3.1) implies* $\operatorname{Re} Q_{N_r} \geq 0$ *(even without property (R)).*

Proof. Decompose $v \in V^{\mathbb{C}}$ as $v = w + \sum\limits_{\lambda \in \mathbb{R} \cap \operatorname{spec} S} v_\lambda$, where $w \in V_i$ and $v_\lambda \in V_\lambda$. Then $N_r v = \sum\limits_\lambda (S - \lambda) v_\lambda$, where $(S-\lambda)v_\lambda \in \operatorname{Ker}(S-\lambda)$, since $N_r^2 = 0$. By Proposition 3.1, we have then $\overline{(S-\lambda)v_\lambda} \in \operatorname{Ker}(S+\lambda) \subset V_{-\lambda}$, so that

$$\begin{aligned}
\sigma(v, \overline{N_r v}) &= \sigma(w + \sum_\mu v_\mu, \sum_\lambda \overline{(S-\lambda)v_\lambda}) \\
&= \sum_{\lambda,\mu} \sigma(v_\mu, \overline{(S-\lambda)}v_\lambda) = \sum_\lambda \sigma(v_\lambda, \overline{(S-\lambda)v_\lambda}),
\end{aligned}$$

since $\sigma(V_r, V_i) = 0$. Moreover, $\operatorname{Re} \sigma(v_\lambda, \overline{(S-\lambda)v_\lambda}) = \operatorname{Re} \sigma(v_\lambda, \overline{Sv_\lambda})$, since $\operatorname{Re} \sigma(z, \overline{z}) = 0 \quad \forall z \in V^{\mathbb{C}}$. We thus obtain

$$\operatorname{Re} \sigma(v_\lambda, \overline{(S-\lambda)v_\lambda}) = \operatorname{Re} \sigma(\overline{v}_\lambda, Sv_\lambda) = \operatorname{Re} Q_S(v_\lambda, \overline{v}_\lambda) \geq 0,$$

and then

$$\operatorname{Re} Q_{N_r}(v, \overline{v}) = \operatorname{Re} Q_{N_r}(\overline{v}, v) = \operatorname{Re} \sigma(\overline{v}, N_r v) = \operatorname{Re}(v, \overline{N_r v}) \geq 0.$$

Q.E.D.

From now on, we assume that $\operatorname{spec} S \subset \mathbb{R}$, i.e. $S = S_r$ and $N = N_r$, and that $N^2 = 0$. By Lemma 3.4, this implies $\operatorname{Re} Q_N \geq 0$. We put

$$W := N_1(V) + N_2(V) = \{\operatorname{Re}(Nz) : z \in V^{\mathbb{C}}\},$$

if $N = N_1 + iN_2$, and

$$K := W^\perp = \{v \in V : \sigma(v, w) = 0 \quad \forall w \in W\}.$$

Proposition 3.5 *If* $\operatorname{spec} S \subset \mathbb{R}, \operatorname{Re} Q_S \geq 0$ *and* $N^2 = 0$, *then W is an isotropic subspace of V, and* $K = \operatorname{Ker} N_1 \cap \operatorname{Ker} N_2 \supset W$.

Moreover, $S_1(K) = 0,\ S_2(K) \subset K,\ S_2(W) \subset W$, *so that in particular also* $D_1(K) = 0,\ D_2(K) \subset K$ *and* $D_2(W) \subset W$.

Finally

$$K^{\mathbb{C}} = \sum_{\lambda \in \operatorname{spec} S} \operatorname{Ker} S_\lambda \tag{3.5}$$

and

$$W = \sum_{\lambda \in \operatorname{spec} S, \lambda \geq 0} W_\lambda, \tag{3.6}$$

where $W_\lambda := W \cap (\operatorname{Ker}(S-\lambda) + \operatorname{Ker}(S+\lambda))$.

Proof. Since $N^2 = (N_1 + iN_2)^2 = 0$, we have

$$N_1^2 = N_2^2 \text{ and } N_1N_2 + N_2N_1 = 0. \tag{3.7}$$

Denote by V^1_μ the generalized eigenspace of N_1 corresponding to $\mu \in \operatorname{spec} N_1 \subset \mathbb{C}$. (3.7) implies that $N_2(V^1_\mu) \subset V^1_{-\mu}$. Moreover, since $Q_{N_1} \geq 0$, all eigenvalues μ of N_1 are purely imaginary. Indeed, if $N_1v = \mu v, v \neq 0$ and $\mu \neq 0$, then $Q_{N_1}(v, \overline{v}) \neq 0$ (since otherwise $N_1v = 0$), and $Q_{N_1}(v, \overline{v}) = \sigma(\overline{v}, N_1v) = \mu\sigma(v, \overline{v})$, where $\sigma(v, \overline{v}) \in i\mathbb{R}$. This implies $\mu \in i\mathbb{R}$. Thus, either $\mu = 0$, or $\mu = i\nu, \nu \in \mathbb{R} \setminus \{0\}$. We shall exclude the second possibility.

Namely, if $\nu \in \mathbb{R} \setminus \{0\}$, then $V^1_{i\nu} \oplus V^1_{-i\nu}$ is an N_1 and N_2-invariant symplectic subspace of $V^\mathbb{C}$, obviously also invariant under complex conjugation. Thus $V^1_{i\nu} \oplus V^1_{-i\nu}$ is the complexification of its intersection with V. We may therefore, for a moment, restrict ourselves to the latter subspace and assume that $V^\mathbb{C} = V^1_{i\nu} \oplus V^1_{-i\nu}$. But, since $Q_{N_1} \geq 0$, we then have in fact $Q_{N_1} > 0$, which means that Q_N satisfies the cone-condition. Consequently, by [5], Lemma 3.2, (see also [11], Lemma 3.1), $\ker N = 0$, a contradiction.

We have shown that N_1 has the only eigenvalue 0, and since $Q_{N_1} \geq 0$, by the classification of normal forms of quadratic forms on symplectic vector spaces (see e.g. [5],Theorem 3.1], we have $N_1^2 = 0$. By (3.7), then also $N_2^2 = 0$.

We will show that indeed

$$N_1^2 = N_2^2 = N_1N_2 = N_2N_1 = 0. \tag{3.8}$$

By (3.7), for $v \in V$ we have

$$\begin{aligned} Q_{N_1}(N_2v, N_2v) &= \sigma(N_2v, N_1N_2v) \\ &= -\sigma(N_2, v, N_2N_1v) = \sigma(N_2^2v, N_1v) = 0, \end{aligned}$$

which implies that $N_1(N_2v) = 0$, since $Q_{N_1} \geq 0$. Consequently, $N_1N_2 = 0$, hence also $N_2N_1 = 0$.

(3.8) shows that $W = N_1(V) + N_2(V)$ is an isotropic subspace of V which is contained in $\operatorname{Ker} N_1 \cap \operatorname{Ker} N_2$. But one sees easily that $\operatorname{Ker} N_1 \cap \operatorname{Ker} N_2 = W^\perp = K$.

Let us decompose $V^\mathbb{C} = \sum_\lambda^\oplus V_\lambda$, where summation is over all $\lambda \in \operatorname{spec} S \subset \mathbb{R}$, and let $v = \sum\limits_\lambda v_\lambda \in V^\mathbb{C}$, with $v_\lambda \in V_\lambda$. In order to prove (3.5), we observe that $K^\mathbb{C} = \operatorname{Ker} N \cap \operatorname{Ker} \overline{N}$, and that $v \in \operatorname{Ker} N$ if and only if $v_\lambda \in \operatorname{Ker}(S - \lambda)$ for every $\lambda \in \operatorname{spec} S$, i.e. $\operatorname{Ker} N = \sum\limits_\lambda \operatorname{Ker}(S - \lambda)$. By (3.3), this space is self-conjugate, so that $\operatorname{Ker} \overline{N} = \overline{\operatorname{Ker} N}$, which shows (3.5). From (3.5) and (3.4) we obtain $S_1(K) = 0$.

Next, for $v_\lambda \in V_\lambda$, we have $Dv_\lambda = \lambda v_\lambda$, hence

$$Nv_\lambda = (S - \lambda)v_\lambda \in \operatorname{Ker}(S - \lambda). \tag{3.9}$$

Together with (3.3), this implies

$$S(Nv_\lambda + \overline{Nv_\lambda}) = \lambda(Nv_\lambda - \overline{Nv_\lambda}),$$

so that $S(\operatorname{Re}(Nv_\lambda)) = i\lambda \operatorname{Im}(Nv_\lambda)$. Consequently, $S_2(\operatorname{Re}(Nv_\lambda)) = \lambda \operatorname{Im}(Nv_\lambda) \in W$, which shows that W is S_2-invariant. Since $S_2 \in \mathfrak{sp}(V, \sigma)$, this implies that also $K = W^\perp$ is S_2-invariant.

Finally, (3.6) is an immediate consequence of (3.9) and (3.3).

Q.E.D.

Corollary 3.6 *Under the hypotheses of Proposition 3.5, we have $S_1^2 = 0$ and $[S_1, S_2] = [D_1, D_2]$.*

Proof. Since $S_1(K) = 0$, i.e. $K \subset \operatorname{Ker} S_1$, we have $S_1(V) = (\operatorname{Ker} S_1)^\perp \subset K^\perp = W \subset K$, so that

$$S_1(K) = 0,\ S_1(V) \subset W. \tag{3.10}$$

This implies in particular $S_1^2 = 0$, and since $N_1(K) = 0,\ N_2(V) \subset W$, also

$$D_1(K) = 0,\ D_1(V) \subset W. \tag{3.11}$$

Consequently, $D_1^2 = 0$ and $N_j D_1 = D_1 N_j = 0$, for $j = 1, 2$. Moreover, $[D, N] = 0$ implies $[D_1, N_1] = [D_2, N_2]$ and $[D_1, N_2] = -[D_2, N_1]$, so that in fact $0 = [D_1, N_1] = [D_2, N_2]$ and $0 = [D_1, N_2] = -[D_2, N_1]$. Since also $[N_1, N_2] = 0$, we find that $[S_1, S_2] = [D_1, D_2]$.

Q.E.D.

Corollary 3.7 *Assume that S satisfies the hypotheses of Proposition 3.5 as well as property (R). Then $[D_1, D_2] = 0$ if and only if $D_1 = 0$.*

Proof. One implication being trivial, we assume that $[D_1, D_2] = 0$. Because of (R), we can decompose V as $V = \sum_{\lambda \neq 0}^{\oplus} (V_\lambda \oplus V_{-\lambda}) \cap V \oplus V_0 \cap V$, where all subspaces in this decomposition are S_1- and S_2-invariant, symplectic and pairwise orthogonal. We may therefore reduce ourselves to one of these spaces, i.e. we may assume that $\operatorname{spec} S = \{-\lambda, \lambda\}$, for some $\lambda \in \mathbb{R}$. The case $\lambda = 0$ being trivial, let $\lambda \neq 0$. Then, $D^2 = \lambda^2 I$, hence

$$D_1^2 - D_2^2 = \lambda^2 I,\ D_1 D_2 + D_2 D_1 = 0.$$

But, from (3.11), we know that $D_1^2 = 0$, and since $D_1 D_2 = D_2 D_1$, we thus find that $D_1 D_2 = D_2 D_1 = 0$ and $D_2^2 = -\lambda^2 I$. This implies $D_1 = 0$.

Q.E.D.

We conclude this section with a result, which shows that the only way that $\operatorname{Re} Q_D$ can be semi-definite is that $D_1 = 0$.

Lemma 3.8 *Let $D = D_1 + iD_2 \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma)$ be such that D is semisimple and $\operatorname{spec} D \subset \mathbb{R}$. Then $\operatorname{Re} Q_D$ is semi-definite if and only if $D_1 = 0$.*

Proof. Let $Q_D = Q_1 + iQ_2$. We first observe that if Q_1 is semi-definite, i.e. if $Q_1 \geq 0$ or $-Q_1 \geq 0$, then $Q_1(z, \overline{z}) = 0$ implies that $Q_1(w, z) = 0$ for every $w \in V^{\mathbb{C}}$, that is, $D_1 z = 0$.

Next, if $0 \neq v \in V_\lambda$, we have $Dv = \lambda v$, hence $Q_1(v, \overline{v}) = \lambda \operatorname{Re} \sigma(v, \overline{v}) = 0$, so that $D_1 v = 0$. We have thus shown that D_1 vanishes on every eigenspace V_λ of D, hence $D_1 = 0$.

Inversely, if $D_1 = 0$, then clearly $\operatorname{Re} Q_D = 0$ is semi-definite.

Q.E.D.

4 Examples.

Before we turn to the proofs of our main theorems, we shall discuss some examples, including the ones from Section 2, in order to illustrate the conditions we imposed in our algebraic results of Section 3.

Our first example demonstrates that the conditions in Theorem 2.7 are weaker than the cone condition.

Example 4.1 On $\mathbb{H}_2$, consider

$$L_S := X_1^2 + X_2^2 + i(X_2Y_2 + Y_2X_2).$$

Obviously, L_S does not satisfy the cone condition. But, $S = -AJ$ has one Jordan block $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and one block $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, since $SY_1 = -X_1$, $SX_1 = 0$, and $SX_2 = iX_2$, $S(Y_2 - \frac{i}{2}X_2) = -i(Y_2 - \frac{i}{2}X_2)$. Thus, $S_r = N_r \neq 0$ and $N_r^2 = 0$.

Example 2.3 The operator in this example can be written as

$$L_S = 2i(X_1Y_2 - (X_2 + iX_3)Y_1) + Y_1^2 + Y_2^2 + X_3^2 + (Y_3 - iY_2)^2.$$

Observe that $\tilde{X}_1 := X_1, \tilde{X}_2 := X_2 + iX_3, \tilde{X}_3 := X_3, \tilde{Y}_1 := Y_1, \tilde{Y}_2 := Y_2, \tilde{Y}_3 := Y_3 - iY_2$ is a complex symplectic basis of $\mathbb{C}^6$, and that L_S can be written

$$L_S = 2i(\tilde{X}_1\tilde{Y}_2 - \tilde{X}_2\tilde{Y}_1) + \tilde{Y}_1^2 + \tilde{Y}_2^2 + \tilde{X}_3^2 + \tilde{Y}_3^2.$$

The matrix $\tilde{S}$ corresponding to the new basis is thus given by

$$\tilde{S} = -\left(\begin{array}{cc|cc} 0 & iJ & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -iJ & I & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & 0 \\ & & 0 & 1 \end{array}\right) \cdot \left(\begin{array}{cc|cc} 0 & I & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -I & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & -1 & 0 \end{array}\right) = \left(\begin{array}{cc|cc} iJ & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ I & iJ & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & -1 \\ & & 1 & 0 \end{array}\right),$$

with respect to the blocks of symplectic coordinates corresponding to $\tilde{X}_1, \tilde{X}_2, \tilde{Y}_1, \tilde{Y}_2$ and $\tilde{X}_3, \tilde{Y}_3$. Here, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The first block of $\tilde{S}$ has eigenvalues ± 1, the second $\pm i$, hence the first corresponds to S_r and the second to S_i. We thus find that $L_S = L_{S_r} + L_{S_i}$, with

$$\begin{aligned} L_{S_r} &= 2i(X_1Y_2 - (X_2 + iX_3)Y_1) + Y_1^2 + Y_2^2 \\ &= Y_1^2 + Y_2^2 + 2X_3Y_1 + 2i(X_1Y_2 - X_2Y_1) \end{aligned}$$

and

$$L_{S_i} = X_3^2 + (Y_3 - iY_2)^2 = X_3^2 + Y_3^2 - Y_2^2 - 2iY_2Y_3.$$

This shows that we have $\operatorname{Re} Q_S \geq 0$, but neither $\operatorname{Re} Q_{S_r} \geq 0$ nor $\operatorname{Re} Q_{S_i} \geq 0$, even though N_r is obviously 2-step nilpotent.

In order to see that L_{S_r} and L_{S_i} satisfy Hörmander's condition (H), observe that for $S', S'' \in \mathfrak{sp}(n, \mathbb{R})$, the Poisson bracket of the principal symbols of $L_{S'}$ and $L_{S''}$ corresponds to the principal symbol of the commutator $[L_{S'}, L_{S''}]$.

And,

$$[L_{\operatorname{Re} S_r}, L_{\operatorname{Im} S_r}] = [Y_1^2 + Y_2^2 + 2X_3Y_1, X_1Y_2 - X_2Y_1] = -2X_3Y_2U,$$

so that, at the origin, for the operator ($L_R +$ first order term), the condition (H) reduces to solving the system

$$\eta_1^2 + \eta_2^2 + 2\xi_3\eta_1 = 0, \quad \xi_1\eta_2 - \xi_2\eta_1 = 0, \quad \xi_3\eta_2 \neq 0.$$

One solution is given by $\xi_1 = \xi_2 = \xi_3 = 1,\ \eta_1 = \eta_2 = \eta_3 = -1.$

Similarly, since $[X_3^2 + Y_3^2 - Y_2^2, Y_2Y_3] = 2X_3Y_2U$, condition (H) for the operator L_{S_i} reduces to solving the system

$$\xi_3^2 + \eta_3^2 - \eta_2^2 = 0, \quad \eta_2\eta_3 = 0, \quad \xi_3\eta_2 \neq 0.$$

A solution is given whenever $\eta_3 = 0$ and $\xi_3 = \eta_2 = 1$.

Example 2.4 For $b \in \mathbb{R} \setminus \{0\}$, consider on $\mathbb{H}_3$ the operator

$$L_S := X_2^2 + X_3^2 + Y_3^2 + 2i(X_1Y_2 + bX_2Y_3).$$

Then, clearly $\operatorname{Re} A \geq 0$, and $S = -AJ$ is given by

$$S = \left(\begin{array}{ccc|ccc} 0 & & & 0 & i & 0 \\ & 1 & & 0 & 0 & ib \\ & & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & & \\ i & 0 & 0 & & 0 & \\ 0 & ib & 0 & & & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right) = \left(\begin{array}{ccc|ccc} 0 & i & 0 & 0 & & \\ 0 & 0 & ib & & -1 & \\ 0 & 0 & 0 & & & -1 \\ \hline 0 & & & 0 & 0 & 0 \\ & 0 & & -i & 0 & 0 \\ & & 1 & 0 & -ib & 0 \end{array}\right).$$

Then, with respect to the new, complex symplectic basis

$$\tilde{X}_1 := Y_1 - bY_3,\ \tilde{X}_2 := -b^2X_1 + bX_3 - iY_2,\ \tilde{Y}_1 := -X_1,\ \tilde{Y}_2 := -iX_2$$

$$\tilde{X}_3 := X_3 + bX_1 - bX_2 + iY_3,\ \tilde{Y}_3 := \frac{i}{2}(X_3 + bX_1 + bX_2 - iY_3),$$

S is represented by the block matrix

$$\tilde{S} = \left(\begin{array}{c|cc} M & \multicolumn{2}{c}{0} \\ \hline \multirow{2}{*}{0} & -i & 0 \\ & 0 & i \end{array}\right),$$

where

$$M := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -b^2 & 0 & 0 & -1 \\ 0 & -(b^2+1) & 0 & 0 \end{pmatrix}.$$

One checks easily that M is 4-step nilpotent, hence conjugate to the matrix

$$\begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \end{pmatrix}.$$

Consequently, $S_r = N_r$ is 4-step nilpotent, and with respect to our complex basis, S_r is represented by the matrix $\tilde{S}_r = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array}\right)$, and $\tilde{S}_i$ by $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} -i & 0 \\ 0 & i \end{matrix} \end{array}\right)$. Since

$$M \cdot \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -b^2 & 0 \\ 0 & 0 & 0 & -(b^2+1) \end{pmatrix} \text{ and } \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \text{ we}$$

thus find that

$$L_{S_r} = 2\tilde{X}_2\tilde{Y}_1 - b^2\tilde{Y}_1^2 - (b^2+1)\tilde{Y}_2^2 = b^2X_1^2 + (b^2+1)X_2^2 - 2bX_1X_3 + 2iX_1Y_2$$

and

$$L_{S_i} = -2i\tilde{X}_3\tilde{Y}_3 = (X_3 + bX_1)^2 - b^2X_2^2 + Y_3^2 + 2ibX_2Y_3.$$

Clearly, neither is $Q_{\mathrm{Re}\, S_r} \geq 0$, nor $Q_{\mathrm{Re}\, S_i} \geq 0$. And, arguing similarly as in the previous example, condition (H) for ($L_{S_r} +$ first order term) reduces to solving the system

$$b^2\xi_1^2 + (b^2+1)\xi_2^2 - 2b\xi_1\xi_3 = 0, \ \xi_1\eta_2 = 0, \ \xi_1\xi_2 \neq 0.$$

A solution is given by $\xi_1 = \xi_2 = 1, \ \xi_3 = (2b^2+1)/(2b), \ \eta_j = 0, \ j = 1, 2, 3.$

In a similar way, one checks that also L_{S_i} satisfies condition (H).

The last two examples show that one can neither dispense with the condition (R), nor with the condition that N_r be nilpotent of step at most two, in Proposition 2.2.

We remark that Example 2.4 is, in a way, of minimal dimension, if one wants to show the latter statement. More precisely, it is of minimal possible dimension, if one requires the nilpotent part N_r to consist of just one Jordan block. This can easily be seen from the classification of normal forms of elements in $\mathfrak{sp}(n, \mathbb{C})$ (see e.g. [5], Theorem 2.1), which reveals that nilpotent elements in $\mathfrak{sp}(n, \mathbb{C})$ consisting of just one Jordan block are nilpotent of even step.

Example 2.6 On $\mathbb{H}_2$, consider

$$L_S := (m + c_1)Y_1^2 + (m - c_1)Y_2^2 + 2c_2Y_1Y_2 + 2i(X_1Y_2 - X_2Y_1).$$

We assume that c_1, c_2 and m are real, that $c_1^2 + c_2^2 \neq 0$ and $m \geq \sqrt{c_1^2 + c_2^2}$. Then A is given by

$$A = \left(\begin{array}{cc|cc} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ \hline 0 & -i & m + c_1 & c_2 \\ i & 0 & c_2 & m - c_1 \end{array}\right),$$

and one verifies readily that $\mathrm{Re}\, A \geq 0$, since $m \geq \sqrt{c_1^2 + c_2^2}$. Moreover, $S = D + N$, where D and N are the block matrices

$$D = \begin{pmatrix} iJ & 0 \\ C & iJ \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ mI & 0 \end{pmatrix},$$

with $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $C := \begin{pmatrix} c_1 & c_2 \\ c_2 & -c_1 \end{pmatrix}$.

One computes that $D^2 = I$, since $CJ + JC = 0$. This implies that D is conjugate to the matrix $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, hence semi-simple. Moreover, N is 2-step nilpotent and commutes with D, so that $S = D + N$ is the Jordan decomposition of S. Clearly, $\operatorname{spec} S = \{-1, 1\}$. But,

$$\operatorname{Re} Q_D(v) = {}^t v \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} v, \quad v \in \mathbb{R}^2 \times \mathbb{R}^2,$$

and $\det C = -(c_1^2 + c_2^2) < 0$, so that $\operatorname{Re} Q_C$ is an indefinite quadratic form. This is in agreement with Lemma 3.8, since D is not purely imaginary.

Remark 4.2 In the study in [13] of operators L_S with real matrices $S \in \mathfrak{sp}(n, \mathbb{R})$ whose spectrum is purely real, it had been most useful to apply the Jordan decomposition $S = D + N$ of S in order to factorize $\Gamma^\mu_{t,S} = \Gamma^\mu_{t,D} \times_\mu \Gamma^\mu_{t,N}$. As the previous example shows, there is no hope of extending this approach to the complex coefficient case, since $\operatorname{Re} Q_D$ may not be positive semi-definite, so that $\Gamma^\mu_{t,D}$ may not be tempered.

5 Twisted convolution and Gaussians generated by $\tilde{L}^\mu_S$.

Assume that $S \in \mathfrak{sp}(n, \mathbb{C})$ is such that $\operatorname{Re} Q_S \geq 0$. It is our main goal in this section to determine the semigroup generated by the operator $|U|^{-1} L_S$. Our results present a generalization of corresponding results in [12],[13], [11], and are directly related to those in [5] by means of the Weyl transform. Instead of transferring the result from [5], Theorem 4.3, by means of the inverse Weyl transform, we prefer, however, to give a direct argument, based on [11], Theorem 5.2 and ideas from [5] and [6].

We shall work in the setting of an arbitrary real symplectic vector space (V, σ) of dimension $2n$. Given two suitable functions φ and ψ on V and $\mu \in \mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, we define the *$\mu$-twisted convolution* of φ and ψ as

$$\varphi \times_\mu \psi(v) := \int_V \varphi(v - v')\psi(v') e^{-\pi i \mu \sigma(v,v')}\, dv',$$

where dv' stands for the volume form $\sigma^{\wedge(n)}$.

If f is a suitable function on $\mathbb{H}_V$, we denote by

$$f^\mu(v) := \int_{-\infty}^{\infty} f(v, u) e^{-2\pi i \mu u} du$$

the partial Fourier transform of f in the central variable u at $\mu \in \mathbb{R}$.

For $\mu \neq 0$, we have

$$(f * g)^\mu(v) = f^\mu \times_\mu g^\mu(v). \tag{5.1}$$

Moreover, if A is any left-invariant differential operator on $\mathbb{H}_V$, then there exists a differential operator $\tilde{A}^\mu$ on V such that

$$(Af)^\mu = \tilde{A}^\mu f^\mu. \tag{5.2}$$

Explicitly, if $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ are coordinates on V associated with a symplectic basis $\{X_j, Y_j\}$, then

$$
\begin{aligned}
\tilde{X}_j^\mu \varphi &= (\partial_{x_j} - \pi i \mu y_j)\varphi = \varphi \times_\mu (\partial_{x_j}\delta_0), \\
\tilde{Y}_j^\mu \varphi &= (\partial_{y_j} + \pi i \mu x_j)\varphi = \varphi \times_\mu (\partial_{y_j}\delta_0) \\
\tilde{U}^\mu \varphi &= 2\pi i \mu \varphi,
\end{aligned}
\tag{5.3}
$$

and consequently,

$$
\tilde{L}_S^\mu = \tilde{\mathcal{L}}_A^\mu = \sum_{j,k} a_{jk} \tilde{V}_j^\mu \tilde{V}_k^\mu
\tag{5.4}
$$

is obtained from $\mathcal{L}_A$ by replacing each V_j in (1.2) by $\tilde{V}_j^\mu$. We remark that, for twisted convolutions with $\partial_w \delta_0$ on the left, there are analogous formulas:

$$
\begin{aligned}
(\partial_{x_j}\delta_0) \times_\mu \varphi &= (\partial_{x_j} + \pi i \mu y_j)\varphi, \\
(\partial_{y_j}\delta_0) \times_\mu \varphi &= (\partial_{y_j} - \pi i \mu x_j)\varphi.
\end{aligned}
\tag{5.5}
$$

On V, we define the *(adapted) Fourier transform* by

$$
\hat{f}(w) := \int_V f(v) e^{-2\pi i \sigma(w,v)} dv, \quad w \in V.
$$

Observe that then $\hat{\hat{f}} = f$ and $\int fg = \int \hat{f}\hat{g}$, for suitable functions f and g on V.

Consider an arbitrary quadratic form Q on $V^{\mathbb{C}}$, with associated Hamilton map $S \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma)$. Once we have fixed a symplectic basis $\{X_j, Y_j\}$ of V, we may identify S with a $2n \times 2n$-matrix. If $\pm\lambda_1, \ldots, \pm\lambda_m$ are the non-zero eigenvalues of S, then $\det(\cos S) = \prod_{j=1}^m \cos^2 \lambda_j$, so that the square root

$$
\sqrt{\det(\cos S)} := \prod_{j=1}^m \cos \lambda_j
\tag{5.6}
$$

is well-defined. Observe that this expression is invariant under all permutations of the roots of the characteristic polynomial $\det(S - \lambda I)$, hence an entire function of the elementary symmetric functions, which are polynomials in (the coefficients of) S.

Thus, as already observed in [5], $\sqrt{\det(\cos S)}$, given by (5.6), is a well-defined analytic function of $S \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma)$.

We shall always consider $T_S := \tilde{L}_S^\mu$ as the maximal operator defined by the differential operator (5.4) on $L^2(V)$; its domain $\mathrm{dom}(T_S)$ consists of all functions $f \in L^2(V)$ such that $\tilde{L}_S^\mu f$, defined in the distributional sense, is in $L^2(V)$.

Lemma 5.1 *$\tilde{L}_S^\mu$ is a closed operator. It is the closure of its restriction to $\mathcal{S}(V)$.*

The proof of Lemma 5.1 will be based on the following well-known observation, which follows easily from the formulas (5.3) and (5.5) (compare also [6]).

Lemma 5.2 *For $w \in V$, denote by ∂_w the directional derivative $\partial_w f(v) = \frac{d}{dt}|_{t=0} f(v + tw)$, and put $\varepsilon_w := \partial_w \delta_0$. Then, the topology in $\mathcal{S}(V)$ is induced by the semi-norms*

$$
||\varepsilon_{w_1} \times_\mu \cdots \times_\mu \varepsilon_{w_N} \times_\mu f \times_\mu \varepsilon_{w_1} \times_\mu \cdots \times_\mu \varepsilon_{w_N}||,
$$

where $w_1, \ldots, w_N, w_1', \ldots, w_N'$ are arbitrary elements of V.

Proof of Lemma 5.1. The continuity of $\tilde{L}^\mu_S$ on $\mathcal{D}'(V)$ implies the closedness of the operator T_S.

Next, observe that if $f \in L^2(V)$ and $\varphi, \psi \in \mathcal{S}(V)$, then it follows readily from Lemma 5.2 that $\varphi \times_\mu f \times_\mu \psi \in \mathcal{S}(V)$.

Choose a Dirac family $\{\varphi_\varepsilon\}$ in $\mathcal{D}(V)$ such that $\varphi_\varepsilon(v) = \varepsilon^{-2n}\varphi(\varepsilon^{-1}v)$, and assume that $f \in \mathrm{dom}(T_S)$. Then $f_\varepsilon := \varphi_\varepsilon \times_\mu f \times_\mu \varphi_\varepsilon \in \mathcal{S}(V)$, and clearly $f_\varepsilon \to f$ in $L^2(V)$ as $\varepsilon \to 0$. Moreover,

$$\tilde{L}^\mu_S f_\varepsilon = \varphi_\varepsilon \times_\mu f \times_\mu (\tilde{L}^\mu_S \varphi_\varepsilon),$$

by the left-invariance of L_S. And, straight-forward computations based on (5.3) – (5.4) and the symmetry of L_S shows that

$$f \times_\mu (\tilde{L}^\mu_S \varphi_\varepsilon) - (\tilde{L}^\mu_S f) \times_\mu \varphi_\varepsilon = f \times_\mu \eta_\varepsilon,$$

where $\eta_\varepsilon(v) = \varepsilon^{-2n}\eta(\varepsilon^{-1}v),\ \eta \in \mathcal{D}(v)$ and $\int \eta\, dv = 0$. This implies

$$\lim_{\varepsilon \to 0} \tilde{L}^\mu_S f_\varepsilon = \lim_{\varepsilon \to 0} \varphi_\varepsilon \times_\mu (\tilde{L}^\mu_S f) \times_\mu \varphi_\varepsilon = \tilde{L}^\mu_S f$$

in $L^2(V)$, and thus $\{f_\varepsilon\}_\varepsilon$ converges in the graph norm to f.

Q.E.D.

Next, obviously the formal adjoint operator of $\tilde{L}^\mu_S$ is given by $\tilde{L}^\mu_{\overline{S}}$. In view of Lemma 5.1, we thus have $(\tilde{L}^\mu_S)^* = \tilde{L}^\mu_{\overline{S}}$ for the adjoint.

Lemma 5.3 *Assume that* $\mathrm{Re}\, Q_S \geq 0$. *Then the operator* $\tilde{\mathcal{L}}^\mu_A = \tilde{L}^\mu_S$ *and its adjoint are dissipative, hence it generates a contraction semigroup* $\exp(t\tilde{L}^\mu_S)$, $t \geq 0$, *on* $L^2(V)$.

Proof. Clearly, for $f \in \mathcal{S}(V)$, we have

$$\begin{aligned} \mathrm{Re}\,(\tilde{\mathcal{L}}^\mu_A f, f) &= -\mathrm{Re} \sum_{j,k} a_{jk}(\tilde{V}^\mu_j f, \tilde{V}^\mu_k f) \\ &= -\mathrm{Re} \int_V a_{jk} g_j(v)\overline{g_k(v)}\, dv \\ &= -\int_V \mathrm{Re}\, Q_S(g_j(v), \overline{g_k(v)})\, dv \leq 0, \end{aligned}$$

if we set $g_j := \tilde{V}^\mu_j f$. This inequality remains true for arbitrary $f \in \mathrm{dom}(\tilde{L}^\mu_A)$, by Lemma 5.1, hence $\tilde{L}^\mu_S$ is dissipative, and the same is true of the adjoint operator $\tilde{L}^\mu_{\overline{S}}$, since $\mathrm{Re}\, Q_S = \mathrm{Re}\, Q_{\overline{S}}$. But then $\tilde{L}^\mu_S$ generates a contraction semigroup (cf. [18]).

Q.E.D.

For the case where $\mathrm{Re}\, Q_S > 0$, an explicit formula for the semigroup $\exp(\frac{t}{|\mu|}\tilde{L}^\mu_S)$ has been given in [11], Theorem 5.2:

Theorem 5.4 *If* $\mathrm{Re}\, Q_S > 0$, *then for* $f \in L^2(V)$

$$\exp(\frac{t}{|\mu|}\tilde{L}^\mu_S) f = f \times_\mu \Gamma^\mu_{t,S}, \quad t \geq 0, \tag{5.7}$$

where, for $t > 0$, $\Gamma^\mu_{t,S}$ *is a Schwartz function whose Fourier transform is given by*

$$\widehat{\Gamma^\mu_{t,S}}(w) = \frac{1}{\sqrt{\det(\cos 2\pi t S)}} e^{-\frac{2\pi}{|\mu|}\sigma(w, \tan(2\pi t S)w)}. \tag{5.8}$$

This result can be extended to the semi-definite case.

Theorem 5.5 *Denote by $\mathfrak{sp}^+(V^{\mathbb{C}},\sigma)$ the cone of all elements $S \in \mathfrak{sp}(V^{\mathbb{C}},\sigma)$ such that $\mathrm{Re}\, Q_S \geq 0$. Then the mapping $S \mapsto \exp(\tilde{L}_S^\mu)f$ is continuous from $\mathfrak{sp}^+(V^{\mathbb{C}},\sigma)$ to $L^2(V)$ (respectively to $\mathcal{S}(V)$), if $f \in L^2(V)$ (respectively if $f \in \mathcal{S}(V)$), and, for $S \in \mathfrak{sp}^+(V^{\mathbb{C}})$ the mapping $t \mapsto \exp(t\tilde{L}_S^\mu)f$ is smooth from $\mathbb{R}_+$ to $\mathcal{S}(V)$, for every $f \in \mathcal{S}(V)$. Moreover, for $t \geq 0$, the operator $\exp(\frac{t}{|\mu|}\tilde{L}_S^\mu)$ is given by (5.7), where $\Gamma_{t,S}^\mu$ is a tempered distribution depending continuously on S, whose Fourier transform is given by (5.8) whenever $\det(\cos(2\pi tS) \neq 0$.*

Proof. In order to simplify the notation, let us assume $\mu = 1$. We then write $\tilde{A}$ instead of $\tilde{A}^1$, if A is a left-invariant differential operator on $\mathbb{H}_V$, and $\varphi \times \psi$ instead of $\varphi \times_1 \psi$.

It is evident from Theorem 5.4 that, if $\mathrm{Re}\, Q_S > 0$ and $f \in \mathcal{S}(V)$, then $f(t) = \exp(t\tilde{L}_S)f$ is a C^∞-function of t and S with values in $\mathcal{S}(V)$, when $t \geq 0$.

If $w = (w_1, \ldots, w_N), w' = (w_1', \ldots, w_N') \in V^N$, we put $f_{w,w'} := \varepsilon_{w_1} \times \cdots \times \varepsilon_{w_N} \times f \times \varepsilon_{w_1} \times \cdots \times \varepsilon_{w_N'}$.

Then

$$\frac{d}{dt} f_{w,w'} = \varepsilon_{w_1} \times \cdots \times \varepsilon_{w_N} \times (\tilde{L}_S f) \times \varepsilon_{w_1'} \times \cdots \times \varepsilon_{w_N'}. \tag{5.9}$$

But, $\tilde{L}_S$ commutes with twisted convolutions on the left. Moreover, if $W \in V$ is regarded as a left-invariant vector field on $\mathbb{H}_V$, then for $j = 1, \ldots, 2n$ we have $WV_j = V_jW + \sigma(W, V_j)U$. This implies

$$WV_jV_k = V_jV_kW + \sigma(W, V_k)V_jU + \sigma(W, V_j)V_kU,$$

hence, by some easy computation,

$$\begin{aligned} WL_S &= L_SW + 2\sigma(W, \sum_{j,k} a_{jk}V_k)V_jU \\ &= L_SW + 2S(W)U. \end{aligned}$$

Taking the partial Fourier transform in the central variable, we obtain

$$(\tilde{L}_S f) \times \varepsilon_W = \tilde{W}(\tilde{L}_S f) = \tilde{L}_S(\tilde{W} f) + 4\pi i \widetilde{S(W)} f,$$

i.e.

$$(\tilde{L}_S f) \times \varepsilon_W = \tilde{L}_S(f \times \varepsilon_W) + 4\pi i f \times \varepsilon_{S(W)}. \tag{5.10}$$

Applying this repeatedly to (5.9), we get

$$\frac{d}{dt} f_{w,w'} = \tilde{L}_S f_{w,w'} + 4\pi i \sum_{j=1}^N f_{w,(w_1', \ldots, S(w_j'), \ldots, w_N')}. \tag{5.11}$$

Since $\tilde{L}_S$ is dissipative, we conclude that

$$\begin{aligned} \frac{d}{dt} \sum_{w,w' \in \{V_1, \ldots, V_{2n}\}^N} ||f_{w,w'}||_2^2 &= 2 \sum_{w,w' \in \{V_1, \ldots, V_{2n}\}^N} \mathrm{Re} \left(\frac{df_{w,w'}}{dt}, f_{w,w'} \right) \\ &\leq C_{N,S} \sum_{w,w' \in \{V_1, \ldots, V_{2n}\}^N} ||f_{w,w'}||^2, \end{aligned}$$

hence

$$||f(t)||^2_{(N)} := \sum_{w,w'\in\{V_1,\dots,V_{2n}\}^N} ||f_{w,w'}||^2 \le e^{tC_{N,S}}||f(0)||_{(N)}, \tag{5.12}$$

where $C_{N,S} \le C_N(1+||S||)$. Notice that, by Lemma 5.2, the semi-norms $||\cdot||_{(N)}$, $N\in\mathbb{N}$, induce the topology on $\mathcal{S}(V)$.

Next, assume that $\operatorname{Re} Q_S \ge 0$ and $\operatorname{Re} Q_{S'} > 0$, and put $h(t) := \exp(t\tilde{L}_S)f - \exp(t\tilde{L}_{S'})f$, where $f\in\mathcal{S}(V)$. Then

$$\begin{aligned}
&\frac{d}{dt}||h(t)||^2\\
&\quad = 2\mathrm{Re}\,(\tilde{L}_S\exp(t\tilde{L}_S)f - \tilde{L}_{S'}\exp(t\tilde{L}_{S'})f, h(t))\\
&\quad = 2\mathrm{Re}\,(\tilde{L}_S h(t), h(t)) + 2\mathrm{Re}\,(\tilde{L}_{S-S'}, \exp(t\tilde{L}_{S'})f, h(t))\\
&\quad \le 2C||S-S'||\;||\exp(t\tilde{L}_{S'})f||_{(N)}||h(t)||,
\end{aligned}$$

for some $n\in\mathbb{N}$ and $C>0$. Thus, because of (5.12),

$$\frac{d}{dt}||h(t)|| \le C||S-S'||e^{tC_{N,S'}}||f||_{(N)},$$

hence, if we assume without loss of generality that $C_{N,S'}\ge 1$,

$$||h(t)|| \le C||S-S'||(e^{tC_{N,S'}}-1)||f||_{(N)}. \tag{5.13}$$

From (5.13) one deduces that $\exp(t\tilde{L}_S)f$ is continuous as a function of S with values in $L^2(V)$, first, for $f\in\mathcal{S}(V)$, but then also for arbitrary $f\in L^2(V)$, by the contraction property. Once this is shown, it follows easily with the aid of (5.12) that $\exp(t\tilde{L}_S)f$ is also continuous as a function of S with values in $\mathcal{S}(V)$, given $f\in\mathcal{S}(V)$. In particular, if $S=\lim S'$, with $\operatorname{Re} Q_{S'}>0$, then $\Gamma^1_{t,S'}$, given by (5.8), converges in $\mathcal{S}'(V)$ towards a tempered distribution $\Gamma^1_{t,S}$, so that (5.7) holds true for arbitrary $S\in\mathfrak{sp}^+(V^{\mathbb{C}},\sigma)$.

Moreover, the dominated convergence theorem shows that also formula (5.8) remains valid whenever $\det(\cos 2\pi tS)\neq 0$. Clearly, also the mapping $S\mapsto\Gamma^1_{t,S}\in\mathcal{S}'(V)$ is continuous.

Finally, since

$$\mathcal{S}(V)\subset\operatorname{dom}(\tilde{L}_S^k)\quad\text{for every } k\in\mathbb{N},$$

it follows easily from (5.9) and (5.12) that the mapping $t\mapsto\exp(t\tilde{L}_S)f$ is smooth from $\mathbb{R}_+$ to $\mathcal{S}(V)$ if $f\in\mathcal{S}(V)$.

Q.E.D.

Observe that Theorem 5.5 implies that

$$\operatorname{Re}\sigma(w,\tan(2\pi tS)w)\ge 0\quad\forall w\in V,\ t\ge 0, \tag{5.14}$$

whenever $\det(\cos 2\pi tS)\neq 0$.

In the coordinates $v=(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$, the symplectic Fourier transform can be written as

$$\hat{f}(w)=\int f(v)e^{-2\pi i\,{}^t wJv}\,dv,$$

and one computes that $(\partial_{v_j}f)^\wedge(w) = -2\pi i(Jw)_j\hat{f}(w)$, $((Jv)_jf)^\wedge(w) = -\frac{1}{2\pi i}\partial_{w_j}\hat{f}(w)$. This shows that $(\tilde{V}_j^\mu f)^\wedge = \hat{V}_j^\mu \hat{f}$, where explicitly

$$\hat{V}_j^\mu = \frac{\mu}{2}\partial_{w_j} - (2\pi i)(Jw)_j, \quad j = 1, \dots, 2n. \tag{5.15}$$

Of course, $(\tilde{L}_S^\mu f)^\wedge =: \hat{L}_S^\mu \hat{f}$, where

$$\hat{L}_S^\mu = \sum_{j,k} a_{jk}\hat{V}_j^\mu \hat{V}_k^\mu. \tag{5.16}$$

If $\operatorname{Re} Q_S \geq 0$, then it follows from Theorem 5.5 that $|\mu|\frac{\partial}{\partial t}\Gamma_{t,S}^\mu = \tilde{L}_S^\mu \Gamma_{t,S}^\mu$ in the sense of distributions. Taking Fourier transforms, it is clear by (5.8) that the corresponding formula for the Fourier transforms will also hold pointwise, i.e.

$$|\mu|\frac{\partial}{\partial t}\widehat{\Gamma_{t,S}^\mu}(w) = \tilde{L}_S^\mu \widehat{\Gamma_{t,S}^\mu}(w) \quad \forall w \in V, \tag{5.17}$$

whenever $\det(\cos 2\pi t S) \neq 0$.

For *arbitrary* $S \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma)$, and complex $t \in \mathbb{C}$, $w \in V^{\mathbb{C}}$, let us *define* $\widehat{\Gamma_{t,S}^\mu}(w)$ by formula (5.8), whenever $\det(\cos 2\pi t S) \neq 0$. Observe that $\widehat{\Gamma_{t,S}^\mu}$ may not be tempered, if $S \notin \mathfrak{sp}^+(V^{\mathbb{C}}, \sigma)$ or $t \notin \mathbb{R}_+$. By analytic extension, formula (5.17) then remains valid, i.e.

$$|\mu|\frac{\partial}{\partial t}\widehat{\Gamma_{t,S}^\mu}(w) = \tilde{L}_S^\mu \widehat{\Gamma_{t,S}^\mu}(w) \quad \forall w \in V^{\mathbb{C}}, t \in \mathbb{C}, S \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma), \tag{5.18}$$

whenever $\det(\cos 2\pi t S) \neq 0$, if we denote by $\frac{\partial}{\partial t}$ and ∂_{w_j} the complex derivatives with respect to $t \in \mathbb{C}$ and the complex variable w_j in (5.15), respectively.

This allows us to introduce complex symplectic changes of coordinates. Let $T = (T_{jk}) \in \mathrm{Sp}\,(n, \mathbb{C})$ be an arbitrary symplectic matrix, and introduce new symplectic coordinates

$$z = Tw \in \mathbb{C}^{2n}, \ w \in \mathbb{R}^{2n}.$$

Since ${}^tTJT = J$, we have

$$Jw = {}^tT(Jz),$$

hence

$$\hat{V}_j^\mu = \frac{\mu}{2}\sum_k T_{kj}\partial_{z_k} - 2\pi i \sum_k T_{kj}(Jz)_k,$$

when acting on holomorphic functions (such as $\widehat{\Gamma_{t,S}^\mu}$). Putting, in analogy with (5.15),

$$Z_k := \frac{\mu}{2}\partial_{z_k} - 2\pi i(Jz)_k, \quad k = 1, \dots 2n,$$

we get

$$\hat{V}_j^\mu = \sum_k T_{kj}Z_k. \tag{5.19}$$

Assume now that we have a splitting of the (complex) symplectic coordinates in two blocks, $z' = (z_1, \dots, z_q; z_{n+1}, \dots z_{n+q})$ and $z'' = (z_{q+1}, \dots, z_n; z_{n+q}, \dots, z_{2n})$, where $1 \leq q < n$. Then, the following lemma is obvious.

Lemma 5.6 *Let $f(z) = f_1(z')f_2(z'')$, where f_1 and f_2 are holomorphic functions. Then, for $k \in \{1, \ldots, q\} \cup \{n+1, \ldots, n+q\}$, $Z_k f_1$ is again a function of z', and*

$$Z_k f(z) = (Z_k f_1)(z') f_2(z'').$$

For $S \in \mathfrak{sp}(V^{\mathbb{C}}, \sigma)$, denote again by $S = S_r + S_i$ the decomposition given by (2.12).

Proposition 5.7 *Assume that* $\det \cos(2\pi t S) \neq 0$. *Then*

$$\widehat{\Gamma^\mu_{t,S}}(w) = \widehat{\Gamma^\mu_{t,S_r}}(w)\widehat{\Gamma^\mu_{t,S_i}}(w), \tag{5.20}$$

and

$$\hat{L}^\mu_{S_r}\widehat{\Gamma^\mu_{t,S}}(w) = (\hat{L}^\mu_{S_r}\widehat{\Gamma^\mu_{t,S_r}})(w)\widehat{\Gamma^\mu_{t,S_i}}(w) = |\mu|(\partial_t\widehat{\Gamma^\mu_{t,S_r}})(w)\widehat{\Gamma^\mu_{t,S_i}}(w), \tag{5.21}$$

if $t \in \mathbb{C}$, $w \in V^{\mathbb{C}}$.

Proof. Choosing real symplectic coordinates, we may assume that $w \in \mathbb{R}^{2n}$. Since S_r and S_i correspond to different sets of Jordan blocks of S, we can choose $T \in \mathrm{Sp}\,(n, \mathbb{C})$ such that

$$\tilde{S} := TST^{-1} = \begin{pmatrix} \tilde{S}_r & 0 \\ 0 & \tilde{S}_i \end{pmatrix}$$

with respect to suitable blocks z' and z'' of complex symplectic coordinates, say $z' \in \mathbb{R}^{2q}$, $z'' \in \mathbb{R}^{2(n-q)}$, where

$$TS_rT^{-1} = \begin{pmatrix} \tilde{S}_r & 0 \\ 0 & 0 \end{pmatrix}, \quad TS_iT^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{S}_i \end{pmatrix}.$$

In the new coordinates $z = Tw$, $\widehat{\Gamma^\mu_{t,S}}$ has the form

$$\begin{aligned} \widehat{\Gamma^\mu_{t,S}} &= \frac{1}{\sqrt{\det(\cos 2\pi t S)}} e^{-\frac{2\pi}{|\mu|}\, {}^t z \cdot J \tan(2\pi t\tilde{S}) \cdot z} \\ &= \widehat{\Gamma^\mu_{t,\tilde{S}_r}}(z')\widehat{\Gamma^\mu_{t,\tilde{S}_i}}(z''), \end{aligned} \tag{5.22}$$

which proves (5.20). Moreover,

$$\hat{L}^\mu_{S_r} = \sum_{j,k} b_{jk}\hat{V}^\mu_j\hat{V}^\mu_k,$$

where $B = (b_{jk}) = S_rJ$, hence by (5.19),

$$\begin{aligned} \hat{L}^\mu_{S_r} &= \sum_{j,k}\sum_{l,m} b_{jk}T_{lj}Z_lT_{mk}Z_m \\ &= \sum_{l,m}(TB\,{}^tT)_{lm}Z_lZ_m = \sum_{l,m}\Big((TS_rT^{-1})J\Big)_{lm}Z_lZ_m. \end{aligned}$$

Putting $C := \tilde{S}_rJ$, which is an $2q \times 2q$-matrix, we find that

$$\hat{L}^\mu_{S_r} = \sum_{j,k=1}^{2q} C_{jk}Z_jZ_k.$$

Formulas (5.21) are now a consequence of (5.22), Lemma 5.6 and (5.18)

Q.E.D.

6 Reduction to Hamiltonians with purely real spectrum

In this section, we shall prove Theorem 2.1. So, assume that $\operatorname{Re} Q_S \geq 0$ and $S_i \neq 0$, and that (R) holds. We begin with the case where $|\operatorname{Re}\alpha| < \nu$. The following proposition will imply Theorem 2.1 (i).

Proposition 6.1 *For $f \in \mathcal{S}(\mathbb{H}_V)$, the integral*

$$\langle K_\alpha, f\rangle := -\int_{-\infty}^{-\infty}\int_0^{+\infty} e^{-2\pi\alpha t\,\operatorname{sgn}\mu}\langle \Gamma^\mu_{t,S}, f^{-\mu}\rangle\, dt\,\frac{d\mu}{|\mu|} \tag{6.1}$$

converges absolutely and defines a tempered distribution K_α for $|\operatorname{Re}\alpha| < \nu$. Moreover, K_α is a fundamental solution for $L_{S,\alpha}$, i.e. $L_{S,\alpha}K_\alpha = \delta_0$.

Here, $\Gamma^\mu_{t,S} \in \mathcal{S}'(V)$ is given by Theorem 5.5.

Proof. Recall that $S = S_r + S_i$, where

$$\operatorname{spec} S_i = \{\pm\omega_1, \ldots, \pm\omega_{n_1}\} \subset \mathbb{C}\setminus\mathbb{R}$$

and $\nu_j = \operatorname{Im}\omega_j > 0,\ j = 1, \ldots, n_1$. Also, $\nu = \sum\limits_{j=1}^{n_1}\nu_j$, $\nu_{min} = \min\limits_{j=1,\ldots,n_1}\nu_j$. Let

$$\operatorname{spec} S_r \setminus \{0\} = \{\pm\lambda_1, \ldots, \pm\lambda_{n_2}\},$$

where $\lambda_k > 0,\ k = 1, \ldots, n_2$. Then, by Theorem 5.5,

$$\widehat{\Gamma^\mu_{\frac{t}{2\pi},S}}(w) = \frac{1}{\sqrt{\det(\cos tS)}}e^{-\frac{2\pi}{|\mu|}\sigma(w,\tan(tS)w)}, \tag{6.2}$$

whenever

$$\sqrt{\det(\cos tS)} = \prod_{k=1}^{n_2}\cos(t\lambda_k)\prod_{j=1}^{n_1}\cos(t\omega_j) \neq 0. \tag{6.3}$$

Thus, potential singularities in (6.2) arize when $t\lambda_k = \frac{\pi}{2} + \ell\pi$ for some $k \in \{1, \ldots, n_2\}$ and $\ell \in \mathbb{Z}$. By means of partial Fourier transforms, we shall show that these points are in fact not singular, if we consider $t \mapsto \widehat{\Gamma^\mu_{\frac{t}{2\pi},S}}$ as a family of distributions.

For any subset $I \subset \{1, \ldots, n_r\}$, denote by $V_I \subset V$ the real subspace

$$V_I := \sum_{k\in I}^{\oplus} V_{\lambda_k} \oplus V_{-\lambda_k}.$$

Then V_I and $V_I^\perp$ are S_1 and S_2-invariant symplectic subspaces. If we choose real symplectic coordinates w' for V_I and w'' for $V_I^\perp$, then S will be represented by a block matrix

$$S = \begin{pmatrix} S_I & 0 \\ 0 & S_{I^\perp} \end{pmatrix}$$

with respect to the coordinates (w', w'') for V, and similarly as in Section 5 we find that

$$\widehat{\Gamma^\mu_{t,S}}(w', w'') = \widehat{\Gamma^\mu_{t,S_I}}(w')\widehat{\Gamma^\mu_{t,S_{I^\perp}}}(w''), \tag{6.4}$$

where

$$\widehat{\Gamma^\mu_{\frac{t}{2\pi},S_I}}(w') = \frac{1}{\prod_{k\in I}\cos(t\lambda_k)} e^{-\frac{2\pi}{|\mu|}\sigma(w',\tan(tS_I)w')}, \tag{6.5}$$

$$\widehat{\Gamma^\mu_{\frac{t}{2\pi},S_{I^\perp}}}(w'') = \frac{1}{\prod_{k\notin I}\cos(t\lambda_k)\prod_j \cos(t\omega_j)} e^{-\frac{2\pi}{|\mu|}\sigma(w'',\tan(tS_{I^\perp})w'')}. \tag{6.6}$$

From (6.5) one computes that

$$\Gamma^\mu_{\frac{t}{2\pi},S_I}(v') = \frac{c|\mu|^{|I|}}{\prod_{k\in I}\sin(t\lambda_k)} e^{-\frac{\pi}{2}|\mu|\sigma(v',\cot(tS_I)v')}, \tag{6.7}$$

where c is a constant of modulus 1 (see e.g. [5], Theorem 7.6.1). This identity holds, unless $t\lambda_k = \ell\pi$ for some $k \in I$ and $\ell \in \mathbb{Z}$. Observe also that all exponentials in (6.5) – (6.7) are bounded by 1.

Lemma 6.2 *There exist a constant $C > 0$ and a Schwartz norm $||\cdot||_{\mathcal{S}}$ on $\mathcal{S}(V)$, such that, for every $0 \le \delta \le 1$,*

$$\left|\langle\Gamma^\mu_{\frac{t}{2\pi},S}, \varphi\rangle\right| |\le C \frac{|\mu|^{\delta n_2}(1+|\mu|^{n_2})^{1-\delta}}{\prod\limits_{k=1}^{n_2}|\sin(t\lambda_j)|^\delta \prod\limits_{j=1}^{n_1}|(\cos(t\omega_j)|} ||\varphi||_{\mathcal{S}}, \quad \forall t \ge 0, \tag{6.8}$$

for every $\varphi \in \mathcal{S}(V)$.

Proof. Given $t \ge 0$, define a subset $I = I_t$ of $\{1, \dots, n_2\}$ as follows: $k \in \{1, \dots, n_2\}$ belongs to I if and only if there exists an $\ell \in \mathbb{Z}$ such that $|t\lambda_k - \frac{\pi}{2} - \ell\pi| \le \frac{\pi}{4}$. Then, for $k \in I$ we have $|\sin(t\lambda_k)| \ge \cos\frac{\pi}{4} > 0$, and if $k \notin I$, then $|\cos(t\lambda_k)| \ge \cos\frac{\pi}{4}$. Since

$$\begin{aligned} \langle\Gamma^\mu_{t,S}, \varphi\rangle &= \langle\widehat{\Gamma^\mu_{t,S}}, \hat{\varphi}\rangle = \langle\widehat{\Gamma^\mu_{t,S_I}} \otimes \widehat{\Gamma^\mu_{t,S_{I^\perp}}}, \hat{\varphi}\rangle \\ &= \langle\Gamma^\mu_{t,S_I} \otimes \widehat{\Gamma^\mu_{t,S_{I^\perp}}}, \hat{\varphi}^I\rangle, \end{aligned} \tag{6.9}$$

where $\hat{\varphi}^I$ denotes the partial Fouriertransform in v', the formulas (6.6) and (6.7) therefore imply

$$\left|\langle\Gamma^\mu_{t,S}, \varphi\rangle\right| \le \frac{C|\mu|^{|I|}}{\prod_j |\cos(t\omega_j)|} ||\hat{\varphi}^I||_1.$$

This gives (6.8) for $\delta = 0$.

On the other hand, choosing $I = \{1, \dots, n_2\}$, we have

$$\left|\langle\Gamma^\mu_{\frac{t}{2\pi},S}, \varphi\rangle\right| \le C \frac{|\mu|^{n_2}}{\prod_{k=1}^{n_2}|\sin(t\lambda_k)|\prod_{j=1}^{n_1}|\cos(t\omega_j)|} ||\varphi||_{\mathcal{S}},$$

which is the case $\delta = 1$, and (6.8) follows immediately from these extreme cases by interpolation.

Q.E.D.

Observe next that for $t > 0$

$$\frac{1}{\cos(t\omega_j)} = \frac{2e^{it\omega_j}}{1+e^{2it\omega_j}} = 2\sum_{m=0}^{\infty}(-1)^m e^{(2m+1)it\omega_j}, \tag{6.10}$$

which implies

$$\frac{1}{\prod_j |\cos t\omega_j|} = O(e^{-t\nu}), \quad t \geq 0. \tag{6.11}$$

The integral in (6.1) can thus be estimated in modulus by

$$C\int_{-\infty}^{\infty}\int_0^{\infty}\frac{e^{-2\pi(\nu-|\mathrm{Re}\,\alpha|)t}}{\prod_{j=1}^{n_2}|\sin(t\lambda_j)|^\delta}\,dt|\mu|^{\delta n_2-1}(1+|\mu|^{n_2})^{1-\delta}||f^{-\mu}||_{\mathcal{S}}\,d\mu,$$

which is convergent if we choose $\delta > 0$ sufficiently small, provided that $|\mathrm{Re}\,\alpha| < \nu$. One also checks easily that $L_{S,\alpha}K_\alpha = \delta_0$ (compare the proof of Theorem 6.1 in [11]). This completes the proof of Proposition 6.1.

Q.E.D.

Remark 6.3 If we argue in a similar way in Example 2.3, by choosing, for a given $t > 0$, either the expression for $\widehat{\Gamma^\mu_{t,S}}$ or for $\Gamma^\mu_{t,S}$ in order to carry out the estimations, we find that the statement of Proposition 6.1 remains true for Example 2.3. This shows that Proposition 6.1 may be true even when property (R) is not satisfied.

In order to prove Theorem 2.1(ii), let us put

$$R := L_{S_r},\ R_\beta := L_{S_r} + i\beta U, \quad \beta \in \mathbb{C}.$$

Assume that $\beta_1, \dots, \beta_N$ are analytic functions of α, and set, for $|\mathrm{Re}\,\alpha| < \nu$,

$$K^N_\alpha := UR_{\beta_N(\alpha)}\dots R_{\beta_1(\alpha)}K_\alpha \in \mathcal{S}'(\mathbb{H}_V). \tag{6.12}$$

Lemma 6.4 *Assume that $\beta_1, \dots, \beta_N$ have been chosen in such a way that the family of tempered distributions K^N_α extends analytically from the strip $|\mathrm{Re}\,\alpha| < \nu$ to the wider strip $|\mathrm{Re}\,\alpha| < M$. Then $L_{S,\alpha}$ is locally solvable for $|\mathrm{Re}\,\alpha| < M$, provided the operators $R_{\beta_j(\alpha)}, j = 1, \dots, N$, are locally solvable.*

Proof. As $[S, S_r] = 0$, all operators $L_{S,\alpha}, R_{\pm\beta_j}$ and U commute. Thus, if $|\mathrm{Re}\,\alpha| < \nu$,

$$\begin{aligned} L_{S,\alpha}K^N_\alpha &= UR_{\beta_N}\dots R_{\beta_1}L_{S,\alpha}K_\alpha \\ &= UR_{\beta_N}\dots R_{\beta_1}\delta_0, \end{aligned} \tag{6.13}$$

where the β_j have to be evaluated at α. By analyticity, this identity remains valid for $|\mathrm{Re}\,\alpha| < M$.

Since $U, R_{\beta_1(\alpha)}, \ldots, R_{\beta_N(\alpha)}$ are locally solvable, this implies local solvability of $L_{S,\alpha}$ (compare the proof of Lemma 7.4 in [11]).

Q.E.D.

Let us examine when the family of distributions K_α^N, $|\mathrm{Re}\,\alpha| < \nu$, can be extended analytically to a wider strip. By (6.1)

$$\langle K_\alpha^N, f\rangle = -2\pi i \int_{-\infty}^{+\infty} \int_0^{+\infty} e^{-2\pi\alpha t \,\mathrm{sgn}\,\mu} \langle \tilde{R}^\mu_{\beta_N} \ldots \tilde{R}^\mu_{\beta_1} \Gamma^\mu_{t,S}, f^{-\mu}\rangle \, dt \,\mathrm{sgn}\,\mu \, d\mu. \tag{6.14}$$

We decompose the integration in μ into $\int_0^{+\infty} d\mu$ and $\int_{-\infty}^0 d\mu$, and denote the corresponding contributions by $\langle K_\alpha^+, f\rangle$ and $\langle K_\alpha^-, f\rangle$, respectively. In the sequel, we shall only regard K_α^+, since the discussion of K_α^- is similar. So, assume that $\mu > 0$. Observe that

$$\langle K_\alpha^+, f\rangle = -2\pi i \int_0^{+\infty} \int_0^{+\infty} e^{-2\pi\alpha t} \langle \hat{R}^\mu_{\beta_N} \ldots \hat{R}^\mu_{\beta_1} \, \widehat{\Gamma^\mu_{t,S}}, \widehat{f^{-\mu}}\rangle \, dt \, d\mu. \tag{6.15}$$

Moreover, by Proposition 5.7,

$$\begin{aligned}
& e^{-2\pi\alpha t} \langle \hat{R}^\mu_{\beta_N} \ldots \hat{R}^\mu_{\beta_1} \widehat{\Gamma^\mu_{t,S}}, \varphi\rangle \\
&\quad = e^{-2\pi\alpha t} \langle \hat{R}^\mu_{\beta_N} \ldots \hat{R}^\mu_{\beta_2} [(\hat{L}^\mu_{S_r} - 2\pi\beta_1\mu)\widehat{\Gamma^\mu_{t,S_r}}] \widehat{\Gamma^\mu_{t,S_i}}, \varphi\rangle \\
&\quad = \mu \langle \hat{R}^\mu_{\beta_N} \ldots \hat{R}^\mu_{\beta_2} [\partial_t (e^{-2\pi\beta_1 t} \widehat{\Gamma^\mu_{t,S_r}})][e^{2\pi(\beta_1 - \alpha)t} \widehat{\Gamma^\mu_{t,S_i}}], \varphi\rangle,
\end{aligned}$$

hence

$$\begin{aligned}
& e^{-2\pi\alpha t} \langle \hat{R}^\mu_{\beta_N} \ldots \hat{R}^\mu_{\beta_1} \widehat{\Gamma^\mu_{t,S}}, \widehat{f^{-\mu}}\rangle \\
&\quad = \mu \frac{d}{dt} \langle \hat{R}^\mu_{\beta_N} \ldots \hat{R}^\mu_{\beta_2} e^{-2\pi\alpha t} \widehat{\Gamma^\mu_{t,S}}, \widehat{f^{-\mu}}\rangle \\
&\qquad - \mu e^{-2\pi\beta_1 t} \langle \hat{R}^\mu_{\beta_N} \ldots \hat{R}^\mu_{\beta_2} (\widehat{\Gamma^\mu_{t,S_r}} \; \partial_t [e^{2\pi(\beta_1-\alpha)t} \widehat{\Gamma^\mu_{t,S_i}}]), \widehat{f^{-\mu}}\rangle.
\end{aligned} \tag{6.16}$$

Inserting this into (6.15) and integrating by parts (observe that all expressions are smooth in $t \geq 0$, by Theorem 5.5), we obtain

$$\begin{aligned}
\langle K_\alpha^+, f\rangle &= 2\pi i \int_0^{+\infty} \langle \hat{R}^\mu_{\beta_N} \ldots \hat{R}^\mu_{\beta_2} \mathbf{1}, \widehat{f^{-\mu}}\rangle \, \mu \, d\mu \\
&+ 2\pi i \int_0^{+\infty} \int_0^{+\infty} e^{-2\pi\beta_1 t} \left\langle \hat{R}^\mu_{\beta_N} \ldots \hat{R}^\mu_{\beta_2} (\widehat{\Gamma^\mu_{t,S_r}} \, \partial_t [e^{2\pi(\beta_1-\alpha)t} \widehat{\Gamma^\mu_{t,S_i}}]), \widehat{f^{-\mu}} \right\rangle dt \mu \, d\mu.
\end{aligned} \tag{6.17}$$

Notice that the boundary term at $t = +\infty$ vanishes, because of (6.8) and (6.11), since $|\mathrm{Re}\,\alpha| < \nu$.

Now, fix some number $M > \nu$, and denote by

$$\varphi(t) := \frac{1}{\sqrt{\det \cos(tS_i)}} = \frac{1}{\prod_{j=1}^{n_1} \cos(t\omega_j)}.$$

By (6.10), $\varphi(t)$ has the series expansion

$$\varphi(t) = 2^{n_1} \sum_{m\in\mathbb{N}^{n_1}} (-1)^{|m|} e^{it\sum_j (2m_j+1)\omega_j}, \quad t > 0.$$

We truncate this series to those m for which $\sum_j (2m_j+1)\nu_j < M$, so that, for $t > 0$,

$$\varphi(t) = 2^{n_1} \sum_{\sum_j (2m_j+1)\nu_j < M} e^{it\sum_j (2m_j+1)\omega_j} + O(e^{-Mt}). \tag{6.18}$$

Observe that $\widehat{\Gamma^\mu_{t,S}} := \widehat{\Gamma^\mu_{t,S_r}}\ \widehat{\Gamma^\mu_{t,S_i}}$, where

$$\widehat{\Gamma^\mu_{\frac{t}{2\pi},S_i}}(w) = \varphi(t) e^{-\frac{2\pi}{|\mu|}\sigma(w,\tan(tS_i)w)}. \tag{6.19}$$

According to (6.18), we split $\varphi(t)$ into a finite sum of terms, and correspondingly the integral (6.15) into a finite sum of integrals.

Arguing as in the proof of Proposition 6.1, the integral containing the remainder term is absolutely convergent for $|\operatorname{Re}\alpha| < M$ and depends analytically on α in this region.

We then only have to discuss the other terms, and to this end, we imagine that $\varphi(t)$ in (6.19) has been replaced by

$$\tilde{\varphi}(t) = e^{it\sum_j (2m_j+1)\omega_j}, \tag{6.20}$$

for some $m \in \mathbb{N}^{n_1}$ such that $\sum_j (2m_j+1)\nu_j < M$, i.e. that $\widehat{\Gamma^\mu_{\frac{t}{2\pi},S_i}}$ has been replaced in (6.16) by $\tilde{\varphi}(t)e^{-\frac{2\pi}{|\mu|}Q_t}$, where

$$Q_t(w) := \sigma(w, \tan(tS_i)w).$$

We choose, with m as in (6.20),

$$\beta_1 := \alpha - \sum_i (2m_j+1)i\omega_j. \tag{6.21}$$

Then $\operatorname{Re}\beta_1 = \operatorname{Re}\alpha - \sum_j (2m_j+1)\nu_j$, and

$$e^{(\beta_1-\alpha)t}\tilde{\varphi}(t)e^{-\frac{2\pi}{\mu}Q_t} = e^{-\frac{2\pi}{\mu}Q_t}.$$

The next lemma can be proved along the same lines as Lemma 6.3 in [11].

Lemma 6.5 *There exist quadratic forms Q_{jk} on V such that*

$$Q_t(w) = \sigma(w, \tan(tS_i)w) = \sum_{j=1}^{n_1}\sum_{k=0}^{\ell} t^k \tan^{(k)}(\omega_j t) Q_{jk}(w),$$

where $\tan^{(k)}$ is the k-th derivative of the tangent function and $\ell+1$ is the dimension of the largest Jordan block of S_i.

We now obtain

$$\frac{\partial Q_t}{\partial t}(w) = \sum_{j=1}^{n_1}\sum_{k=0}^{\ell} t^k \tan^{(k+1)}(\omega_j t)\tilde{Q}_{jk}(w),$$

for some other quadratic forms $\tilde{Q}_{jk}$. Since

$$\partial_t(e^{-\frac{2\pi}{\mu}Q_t(w)}) = -\frac{2\pi}{\mu}\,\frac{\partial Q_t}{\partial t}(w)e^{-\frac{2\pi}{\mu}Q_t(w)},$$

the second term in (6.16) then decomposes as a finite sum of terms of the form

$$\begin{aligned} c\int_0^{+\infty}\int_0^{+\infty} \quad & e^{-2\pi\beta_1 t}\tan^{(k+1)}(2\pi\omega_{j_0}t)t^k \\ & \cdot\left\langle \hat{R}^{\mu}_{\beta_N}\dots\hat{R}^{\mu}_{\beta_2}\left(\tilde{Q}_{j_0k}\widehat{\Gamma^{\mu}_{t,S_r}}e^{-\frac{2\pi}{\mu}Q_t}\right), \widehat{f^{-\mu}}\right\rangle\, dt\, d\mu. \end{aligned} \tag{6.22}$$

It is now important to make the following observation:

If we choose complex symplectic coordinates z' and z'' corresponding to Jordan blocks of S_r and S_i, respectively, as in the proof of Proposition 5.7, then $\widehat{\Gamma^{\mu}_{t,S_r}}$ depends only on z', Q_t and $\tilde{Q}_{jk}$ on z'', and $\hat{R}^{\mu}_{\beta_j} = L_{S_r} - 2\pi\beta_j\mu$ on z'. Therefore,

$$\begin{aligned} &\left\langle \hat{R}^{\mu}_{\beta_N}\dots\hat{R}^{\mu}_{\beta_2}(\tilde{Q}_{j_0k}\widehat{\Gamma^{\mu}_{t,S_r}}e^{-\frac{2\pi}{\mu}Q_t}),\ \widehat{f^{-\mu}}\right\rangle \\ &\qquad = \frac{1}{\varphi(2\pi t)}\left\langle \tilde{Q}_{j_0l}\hat{R}^{\mu}_{\beta_N}\dots\hat{R}^{\mu}_{\beta_3}([(\hat{L}^{\mu}_{S_r}-2\pi\beta_2\mu)\widehat{\Gamma^{\mu}_{t,S_r}}]\widehat{\Gamma^{\mu}_{t,S_i}}), \widehat{f^{-\mu}}\right\rangle. \end{aligned}$$

Moreover,

$$\tan(\omega_j t) = i - 2i\sum_{p=0}^{\infty}(-1)^p e^{2i(p+1)\omega_j t}, \quad t>0, \tag{6.23}$$

so that

$$\tan^{(k+1)}(\omega_j t) = -2i\sum_{p=0}^{\infty}(-1)^p(2i(p+1))^{k+1}e^{2i(p+1)\omega_j t}.$$

This shows that

$$\cos(\omega_{j_0}t)\tan^{(k+1)}(\omega_{j_0}t) = \sum_{p=0}^{\infty} a_p e^{it(2p+1)\omega_{j_0}t}, \quad t>0 \tag{6.24}$$

and then

$$e^{-2\pi\beta_1 t}\frac{1}{\varphi(2\pi t)}\tan^{(k+1)}(2\pi\omega_{j_0}t) = e^{-2\pi(\alpha-2i\omega_{j_0}t)}\sum_{q\in\mathbb{N}^n} b_q e^{2\pi it\,2q_j\omega_j t},$$

where the latter series converges locally absolutely and uniformly for $t>0$.

If we truncate this series in a similar way as before, such that the remainder term is analytic in $|\mathrm{Re}\,\alpha| < M$, and put everything together, we see that each term (6.22) can be further decomposed into terms which, except for the remainder term, are of the form

$$\begin{aligned} c\int_0^{+\infty}\int_0^{+\infty} & e^{-2\pi(\alpha-\sum_j 2q_j i\omega_j)t}\,t^k \\ & \cdot\left\langle \tilde{Q}_{j_0k}\hat{R}^{\mu}_{\beta_N}\dots\hat{R}^{\mu}_{\beta_3}([\hat{R}^{\mu}_{\beta_2}\widehat{\Gamma^{\mu}_{t,S_r}}]\widehat{\Gamma^{\mu}_{t,S_i}}, \widehat{f^{-\mu}}\right\rangle\, dt\, d\mu, \end{aligned} \tag{6.25}$$

where $q = (q_1, \ldots, q_{n_1}) \in \mathbb{N}^{n_1}$ and $q_{j_0} \geq 1$, i.e. $|q| \geq 1$.

By Lemma 6.2, this integral converges absolutely for $\mathrm{Re}\,\alpha > -(\nu + 2\nu_{j_0} + \sum_j 2q_j\nu_j)$ and not only for $\mathrm{Re}\,\alpha > -\nu$. Thus, the contribution to K_α^N given by the integration over $\mu > 0$ has been extended as an analytic family of distributions from $\mathrm{Re}\,\alpha > -\nu$ to $\mathrm{Re}\,\alpha > -\nu - 2\nu_{\min}$.

We can at this point iterate the argument above, in order to extend K_α^+ to the domain $\mathrm{Re}\,\alpha > -M$. If one compares (6.25) with (6.15), one finds that the only new features are the presence of the quadratic forms $\tilde{Q}_{j,k}$ and the powers t^k of t. The factors $\tilde{Q}_{j,k}$ are harmless, since they only depend on z'', so that the multiplication with $\tilde{Q}_{j,k}$ commutes with each of the operators $\hat{R}^\mu_{\beta_j}$.

As for powers t^k of t, observe that

$$t^k (\hat{R}^\mu_{\beta_2})^{k+1} \widehat{\Gamma^\mu_{t,S_r}} \tag{6.26}$$

$$= \mu\, e^{2\pi\beta_2 t} \frac{\partial}{\partial t} \left[\sum_{j=0}^{k} (-1)^j \frac{k!}{(k-j)!} (t\hat{R}^\mu_{\beta_2})^{k-j} (e^{-2\pi\beta_j t} \widehat{\Gamma^\mu_{t,S_r}}) \right], \tag{6.27}$$

which follows easily from

$$\mu\, e^{2\pi\beta_2 t} \frac{\partial}{\partial t} \left(e^{-2\pi\beta_2 t} \widehat{\Gamma^\mu_{t,S_r}} \right) = \hat{R}^\mu_{\beta_2} \widehat{\Gamma^\mu_{t,S_r}}.$$

Choosing $\beta_j = \beta_2$ for $j = 2, \ldots, k+2$, we may then take (6.24) as a substitute for the latter identity in order to perform the integration by parts argument. Choosing β_2 appropriately, again of the form

$$\beta_2 = \alpha - \sum_j (2m'_j + 1) i\omega_j,$$

we find that, except for some trivial remainder terms, K_α^+ can be written as a finite sum of terms like those in (6.23), only with $\tilde{Q}_{j_0,k}$ replaced by some polynomial in z'' of higher degree, and this time with $|q| \geq 2$.

We can proceed in this way, and find that in the k-th iteration, the β_ℓ's which have to be chosen are of the form

$$\beta_\ell = \alpha - \sum_j (2m_j + 1) i\omega_j,$$

with $\sum_j (2m_j + 1)\nu_j < M$ (otherwise, no further integration by parts will be needed). This is true for K_α^+.

In the discussion of K_α^-, one finds in a similar way that the β_ℓ's will be of the form $\alpha + \sum_j (2m_j + 1) i\omega_j$, again with $\sum_j (2m_j + 1)\nu_j < M$. Consequently, by Lemma 6.4, $L_{S,\alpha}$ is locally solvable, provided $L_{S_r} + i(\alpha \pm \sum_j (2k_j + 1) i\omega_j) U$ is locally solvable whenever $\sum_j (2k_j + 1)\nu_j < M$.

This implies Theorem 2.1 (ii) and completes the proof of Theorem 2.1.

7 Partial results on the case of Hamiltonians with real spectrum

Our main goal in this section will be to give a proof of Theorem 2.5. Observe that this theorem, in combination with Theorem 2.1 and Proposition 2.2, immediately also implies Theorem 2.7

Let us thus assume that $S \neq 0$ has purely real spectrum, that $\operatorname{Re} Q_S \geq 0$, $N^2 = 0$ and that property (C) is satisfied. Then, by Corollary 3.6, we have $[S_1, S_2] = 0$, hence $[L_{S_1}, L_{S_2}] = -2L_{[S_1,S_2]}U = 0$. If we write $\alpha = a + ib$, with a, b real, we therefore obtain

$$L_S + i\alpha U = A + iB,$$

where $A := L_{S_1} + iaU$ and $B := L_{S_2} + ibU$ are formally self-adjoint and commute.

Proposition 7.1 *Assume that A and B are left-invariant differential operators on a Lie group G which are formally self-adjoint on $C_0^\infty(C_1) \subset L^2(G, dg)$, where dg denotes a right-invariant Haar measure on G. If A and B commute, then $A+iB$ is locally solvable, provided either A or B is locally solvable.*

Proof. It is well-known (compare [4], Lemma 6.1.2) that a differential operator L with smooth coefficients on $\mathbb{R}^d$ is locally solvable at x_0 if and only if there exist an open neighborhood U of x_0 and constants $k \in \mathbb{N}, C \geq 0$ such that

$$||\varphi||_{(-k)} \leq C||L^*\varphi||_{(k)} \quad \forall \varphi \in C_0^\infty(U). \tag{7.1}$$

Here, $|| \cdot ||_{(\ell)}$ denotes the Sobolev norm of order ℓ. If L is a left-invariant differential operator on G, then the characterization of local solvability given by (7.1) remains true, if we assume that U is taken so small that it can be covered by a single chart, and if we then define the Sobolev norms by means of the local coordinates, with Lebesgue measure replaced by Haar measure. For given $k \in \mathbb{N}$, we can then find an elliptic right-invariant differential operator Q on G such that $||\psi||_{(k)} \leq ||Q\psi||$, $\psi \in C_0^\infty(U)$, hence (7.1) is equivalent to

$$||\varphi||_{(-k)} \leq C||Q(L^*\varphi)||, \quad \forall \varphi \in C_0^\infty(U). \tag{7.2}$$

Notice that $QL^* = L^*Q$, since L^* is left-invariant and Q is right-invariant. Assume now that A and B satisfy the hypotheses of Proposition 7.1, and suppose for instance that A is locally solvable. Then $||A \pm iB)\psi||^2 = ||A\psi||^2 + ||B\psi||^2$ for every $\psi \in C_0^\infty(G)$, and we may assume that (7.2) is satisfied for $L = A$. This implies

$$\begin{aligned} ||\varphi||_{(-k)} &\leq C||A(Q\varphi)|| \leq C||(A - iB)(Q\varphi)|| \\ &= C||Q((A + iB)^*\varphi)||, \quad \forall \varphi \in C_0^\infty(U), \end{aligned}$$

and consequently also $A + iB$ is locally solvable.

Q.E.D.

Proof of Theorem 2.5. Consider $A = L_{S_1} + iaU$. As $S_1^2 = 0$, by Corollary 3.6, the main theorem in [13] shows that A is locally solvable, unless $S_1 = 0$ and $a = 0$. This implies Theorem 2.5(i), in view of Proposition 7.1.

We have thus reduced ourselves to the case where $A = 0$, i.e. where $L_S + i\alpha U = L_{S_2} + ibU$. But, L_{S_2} is a real-coefficient operator, and so the remaining cases (ii), (iii) in Theorem 2.5 are immediate consequences of [13].

Q.E.D.

It is perhaps interesting to observe the following corollary to Proposition 3.5, which in the case $D = 0$ opens up a different approach to Theorem 2.5.

Proposition 7.2 *Assume that $S^2 = 0$ and $\operatorname{Re} Q_S \geq 0$. Then we can select a symplectic basis $X_1, \dots, X_n,\ Y_1, \dots, Y_n$ of V such that $L_S = \sum_{j,k=1}^{m} b_{jk} Y_j Y_k$ for some $m \leq n$ and $b_{jk} \in \mathbb{C}$.*

Proof. Define the subspaces W and K as in Proposition 3.5, and choose a basis $Y_1, \dots, Y_m$ of the isotropic subspace W. Pick $X_1 \in V$ such that $\sigma(X_1, Y_j) = \delta_{1,j}, j = 1, \dots, m$, and then X_2 such that $\sigma(X_2, Y_2) = 1$ and $X_2 \perp \operatorname{span}\{X_1, Y_1, Y_3, \dots, Y_m\}$, and continue in this way to select X_j's. In the m-th step, this means that we pick X_m such that $\sigma(X_m, Y_m) = 1$ and $X_m \perp \operatorname{span}\{X_1, \dots, X_{m-1}, Y_1, \dots, Y_{m-1}\}$. Then $U := \operatorname{span}\{X_1, \dots, X_m\}$ is an isotropic subspace too, and W and U are in duality with respect to σ. In particular, $U \cap W^\perp = 0$, and

$$V = U \oplus W \oplus H = U \oplus K,$$

where $H := (W \oplus U)^\perp \subset K$ is a symplectic subspace. Moreover, since $S_j = N_j, j = 1, 2$, by the definition of W and K we have

$$S_j(U) \subset W,\ S_j(K) = 0, \quad j = 1, 2.$$

Choose a symplectic basis $X_{m+1}, \dots, X_n, Y_{m+1}, \dots, Y_n$ of H. Then, with respect to the sets of basis elements $X_1, \dots, X_m, Y_1, \dots, Y_m$ and $X_{m+1}, \dots, X_n, Y_{m+1}, \dots, Y_n$, the linear mapping $S = S_1 + iS_2$ is represented by a block matrix of the form

$$S = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

hence $A = S \left(\begin{array}{cc|cc} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \end{array}\right)$ by $A = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$.
This implies the proposition.

Q.E.D.

Observe that, in suitable coordinates, the operator $L_S + i\alpha U$, with S as in Proposition 7.2, is a constant coefficient operator, hence locally solvable, by the Malgrange-Ehrenpreis theorem.

An example which cannot be treated by means of Proposition 7.1 is the operator L_S from Example 2.6. Nevertheless, as mentioned in Section 2, the following proposition holds true.

Proposition 7.3 *Assume that $0 < \sqrt{c_1^2 + c_2^2} \leq m$. Then the operator $L_S + i\alpha U$, with L_S given as in Example 2.6, is locally solvable for every $\alpha \in \mathbb{C}$, even though property (C) is not satisfied.*

Proof. The proof will be based on the representation theoretic criterion given by Theorem 4.1 in [8]. Adapting the notation from [8], denote for $\mu \in \mathbb{R} \setminus \{0\}$ by π_μ the Schrödinger representation of $\mathbb{H}_2$ on $L^2(\mathbb{R}^2)$, with parameter μ. For its differential, we have

$$d\pi_\mu(X_j) = \frac{\partial}{\partial x_j},\ d\pi_\mu(Y_j) = i\mu x_j,\ (j = 1, 2), \quad d\pi_\mu(U) = i\mu,$$

if we denote the coordinates in $\mathbb{R}^2$ by $x = (x_1, x_2)$. Putting $\mathcal{L} := L_S + i\alpha U$, then

$$\begin{aligned}(7.3)\qquad & -d\pi_\mu(\mathcal{L}) \\ & = \mu^2((m+c_1)x_1^2 + (m-c_1)x_2^2 + 2c_2x_1x_2) + 2\mu(x_2\partial_{x_1} - x_1\partial_{x_2}) - \alpha\mu.\end{aligned}$$

Introducing polar coordinates $x_1 = r\cos\theta, x_2 = r\sin\theta,\ r \geq 0$, we obtain

$$d\pi_\mu(\mathcal{L}) = \mu[2\partial_\theta + \alpha - \mu r^2 q(\theta)], \tag{7.4}$$

where $q(\theta) := m + c_1\cos(2\theta) + c_2\sin(2\theta)$. Observe that $q \geq 0$, since $m \geq \sqrt{c_1^2 + c_2^2}$. For θ_0 fixed, we put

$$Q_{\theta_0}(\theta) := \int_{\theta_0}^{\theta} q(\theta)\, d\theta = m(\theta - \theta_0) + \psi(\theta - \psi(\theta_0),$$

where $\psi(\theta) := \frac{c_1}{2}\sin(2\theta) - \frac{c_2}{2}\cos(2\theta)$. Notice that

$$Q_{\theta_0}(\theta) \geq 0, \qquad \text{if } \theta \geq \theta_0, \tag{7.5}$$

and that ψ is 2π-periodic. Define $F^\mu_{\theta_0}(r,\theta)$ to be the 2π-periodic extension of $\tilde{F}^\mu_{\theta_0}(r,\theta)$ given by

$$\tilde{F}^\mu_{\theta_0}(r,\theta) := e^{-\frac{\alpha}{2}(\theta-\theta_0)} e^{\frac{1}{2}\mu r^2 Q_{\theta_0}(\theta)}, \quad \theta_0 \leq \theta < \theta_0 + 2\pi. \tag{7.6}$$

Then one checks easily that, in the sense of (2π-periodic) distributions,

$$\begin{aligned} & d\pi\mu(\mathcal{L})[h(r)F^\mu_{\theta_0}(r,\theta)] \\ & \quad = 2\mu h(r)[F^\mu_{\theta_0}(r,\theta_0) - F^\mu_{\theta_0}(r,\theta_0 + 2\pi)]\delta_{\theta_0}(\theta) \\ & \quad = 2\mu h(r)\ [1 - e^{-\pi\alpha + \pi\mu r^2 m}]\delta_{\theta_0}(\theta).\end{aligned}$$

Notice that for $v = (r_0\cos\theta_0, r_0\sin\theta_0) \neq 0$, we have $\delta_v(x) = \frac{1}{r_0}\delta_{r_0}(r)\delta_{\theta_0}(\theta)$. Therefore, if we put

$$H_\mu(x,v) := \frac{F^\mu_{\theta_0}(r_0,\theta)}{2\mu r_0[1 - e^{-\pi\alpha + \pi\mu r_0^2 m}]}\delta_{r_0}(r), \tag{7.7}$$

we have $H_\mu(\cdot, v) \in \mathcal{S}'(\mathbb{R}^2)$, for a.e. v, and $d\pi_\mu(\mathcal{L})H_\mu(\cdot,v) = \delta(v - \cdot)$. Write $H_\mu = G_\mu\delta_{r_0}(r)$, and $\alpha = a + ib,\ a, b \in \mathbb{R}$. We claim that

$$G_\mu(x,v) = \begin{cases} \frac{g_\mu(x,v)}{\mu r_0}, & \text{if } b \notin 2\mathbb{Z}, \\ \frac{g_\mu(x,v)}{\mu r_0(a - \mu m r_0^2)}, & \text{if } b \in 2\mathbb{Z}, \end{cases} \tag{7.8}$$

where

$$||g_\mu||_\infty \leq C_\alpha, \tag{7.9}$$

with C_α independent of μ.

Indeed, if $\mu < 0$, then $|F^\mu_{\theta_0}| \leq C_\alpha$, since $Q_{\theta_0} \geq 0$ by (7.5). Moreover, if $b \notin 2\mathbb{Z}$, then $|1 - e^{-\pi\alpha + \pi\mu r_0^2 m}|$ is bounded from below, and if $b \in 2\mathbb{Z}$, then $1 - e^{-\pi\alpha + \pi\mu r_0^2 m} = 1 - e^{-\pi(a - \mu m r_0^2)}$, and (7.8), (7.9) follow immediately.

On the other hand, if $\mu > 0$, then for $\mu r_0^2 m >> 1$ we have

$$|\mu r_0 G_\mu(x,v)| \leq C_\alpha e^{\frac{\mu}{2} r_0^2 Q_{\theta_0}(\theta) - \pi\mu r_0^2 m}.$$

Moreover, $Q_{\theta_0}(\theta) \leq Q_{\theta_0}(\theta_0 + 2\pi) = 2\pi m$, and thus $|\mu r_0 G_\mu(x,v)| \leq C_\alpha$. Finally, the case where $\mu r_0^2 m \leq C$ is easy, and again we obtain (7.8), (7.9).

(7.8) shows that singularities may arize, if $b \in 2\mathbb{Z}$, and even if $b \notin 2\mathbb{Z}$, it turns out that some negative powers of μ may arize in a later step of the proof. For given $N \in \mathbb{N}$, we therefore define

$$G_\mu^N(x,v) := \begin{cases} (i\mu)^N G_\mu(x,v), & \text{if } b \notin 2\mathbb{Z}, \\ (i\mu)^N (i\mu a - i\mu^2 m r_0^2), & \text{if } b \in 2\mathbb{Z}, \end{cases}$$

and $H_\mu^N = G_\mu^N \delta_{r_0}(r)$. Then

$$\begin{aligned} d\pi_\mu(\mathcal{L}) H_\mu^N(\cdot, v) &= (i\mu)^N (i\mu a - i\mu^2 m |v|^2)^\sigma \delta(v - \cdot) \\ &= d\pi_\mu(U^N(aU + im(Y_1^2 + Y_2^2))^\sigma)\delta(v - \cdot), \end{aligned} \tag{7.10}$$

where $\sigma := 0$, if $b \notin 2\mathbb{Z}$, and $\sigma = 1$, if $b \in 2\mathbb{Z}$. For $\varphi \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, it then follows easily from (7.8) that the mapping $v \mapsto \langle H_\mu^N(\cdot, v), \varphi(\cdot, v)\rangle$, which, in polar coordinates, is given by

$$\int_0^{2\pi} G_\mu^N((r_0,\theta),(r_0,\theta_0))\, \varphi((r_0,\theta),(r_0,\theta_0))\, r_0\, d\theta,$$

is integrable with respect to v and defines a tempered distribution H_μ^N, namely

$$\begin{aligned} &\langle H_\mu^N, \varphi\rangle \\ &:= \int \langle H_\mu^N(\cdot, v), \varphi(\cdot, v)\rangle dv = \int_0^{+\infty} \int_0^{2\pi} \int_0^{2\pi} G_\mu^N((r_0,\theta),(r_0,\theta_0))\varphi((r_0,\theta))\, d\theta\, d\theta_0 r_0^2\, dr, \end{aligned}$$

for every $\mu \neq 0$. We set $\tilde{H}_\mu^N(x,\eta) := H_\mu^N(\frac{\eta}{\mu} + \frac{x}{2}, \frac{\eta}{\mu} - \frac{x}{2})$, where the change of variables is to be understood in the sense of distributions, i.e.

$$\langle \tilde{H}_\mu^N, \varphi\rangle = \langle H_\mu^N, \varphi(u - v, \frac{\mu}{2}(u+v))\rangle |\mu|^2.$$

Observe that, by (7.8), (7.9), we can find a continuous Schwartz norm $||\cdot||_{\mathcal{S}}$ on $\mathcal{S}(\mathbb{R}^2 \times \mathbb{R}^2)$ and $M \in \mathbb{N}$ such that

$$|\langle H_\mu^N, \psi\rangle| \leq |\mu|^N (|\mu|^M + |\mu|^{-M}) ||\psi||_{\mathcal{S}}, \quad \psi \in \mathcal{S},$$

for every N. This shows that

$$\langle \tilde{H}^N, f\rangle := \int\limits_{\mathbb{R}\setminus\{0\}} \langle \tilde{H}_\mu^N, \psi(\cdot, \mu)\rangle\, d\mu, \quad f \in \mathcal{S}(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R})$$

defines a tempered distribution on $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$, provided we choose N sufficiently large. But then the H_μ^N satisfy the hypothesis of [8], Theorem 4.1, except for condition (i) in this theorem, which has to be replaced by (7.10). The proof of Theorem 4.1 in [8] then still applies and shows that there is a tempered distribution $K^N \in \mathcal{S}'(\mathbb{H}_2)$ such that

$$\mathcal{L} K^N = c U^N (aU + im(Y_1^2 + Y_2^2))^\sigma \delta_0,$$

for a suitable constant $c \neq 0$.

Since the operator $U^N(aU + im(Y_1^2 + Y_2^2))^\sigma$ is locally solvable, this implies local solvability of $\mathcal{L}$ (compare e.g. the proof of Lemma 7.4 in [11]).

Q.E.D.

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